General covariance in effective quantum black hole models

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In collaboration with J. Lewandowski, Y. Ma, J. Yang, Z. Cao

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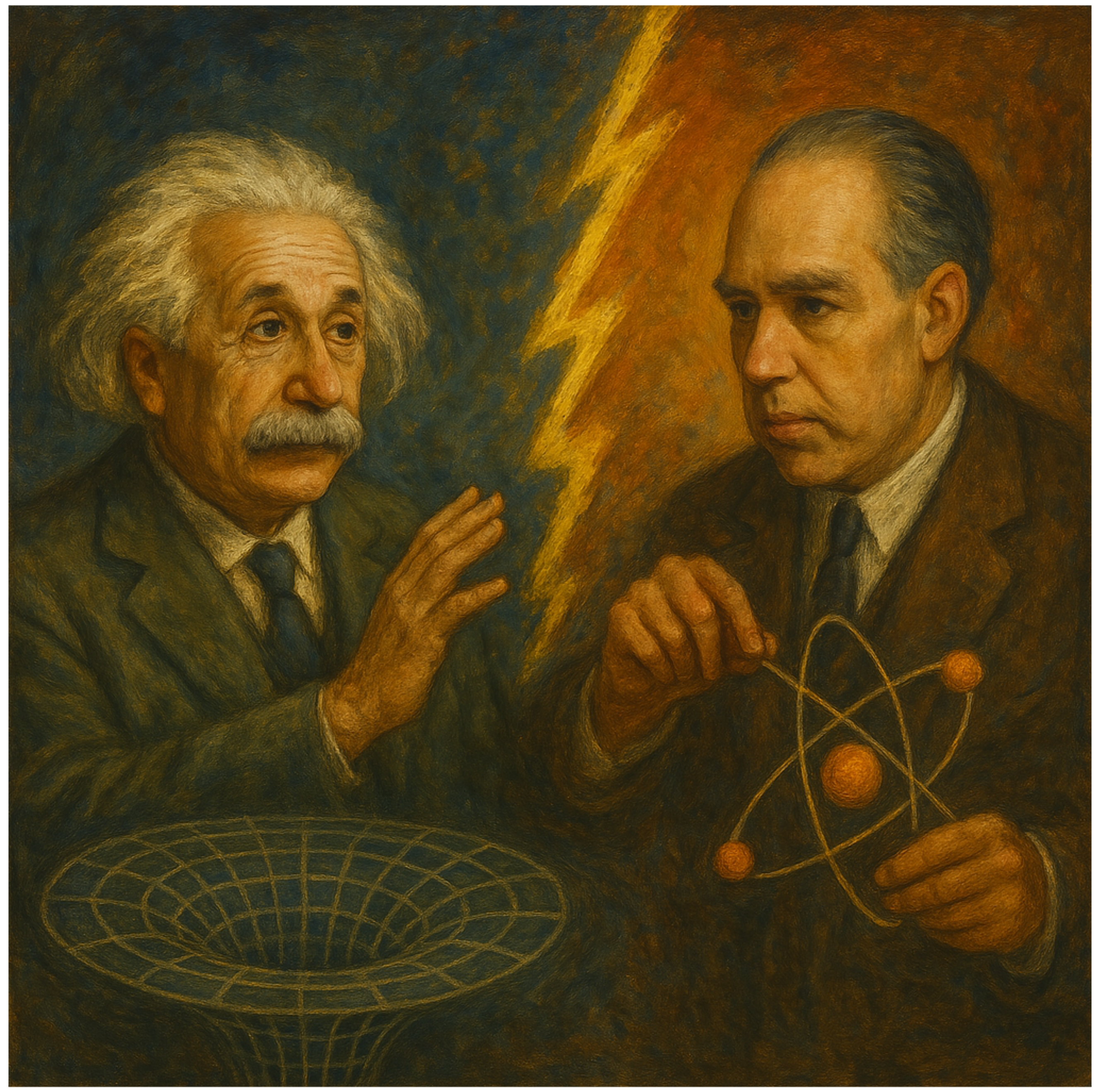
Motivation



Ununification between GR and QM

- **GR is not the final theory on spacetime;**
- **Quantum gravity to unify of GR and QM;**
- **Effective approach to QG:**
  - Spacetime is described by $g_{\mu\nu}$
  - EOM is modified to $G^{\rm eff}_{\mu\nu} = 0$

Motivation

Canonical quantum gravity

↓

Hamiltonian formulation **V.S.** **General covariance**

Space+time **Spacetime**

- **The requirement of a 3+1 decomposition may potentially obscure general covariance.**
- **How can general covariance be restored in the Hamiltonian framework?**

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Canonical quantum gravity

Hamiltonian formulation **V.S.** **General covariance**

Space+time **Spacetime**

- **The requirement of a 3+1 decomposition may potentially obscure general covariance.**
- **How can general covariance be restored in the Hamiltonian framework?**

Investigating this issue in the spherically symmetric gravitation model with $\Sigma \ni (x, \theta, \phi)$

Mathematical settings: spherically symmetric model

- **Phase space:** $(K_I(x), E^I(x)),\ I = 1,\ 2$**;**
- **Dynamics: the Diff constraints** H_x **and the Ham. constraint** H_{eff}**.**

Mathematical settings: spherically symmetric model

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- **Dynamics: the Diff constraints H_x and the Ham. constraint $H_{\rm eff}$.**
  - H_x **is assumed to keep its classical expression, but** $H_{\rm eff}$ **is unknown**
  - **Assume the following constraint algebra**

$$
\begin{aligned}
\{H_x[N_1^x], H_x[N_2^x]\} &= H_x[N_1^x\partial_x N_2^x - N_2^x\partial_x N_1^x], \\
\{H_{\rm eff}[N], H_x[M^x]\} &= -H_{\rm eff}[M^x\partial_x N], \\
\{H_{\rm eff}[N_1], H_{\rm eff}[N_2]\} &= H_x[\mu E^1(E^2)^{-2}(N_1\partial_x N_2 - N_2\partial_x N_1)]\,.
\end{aligned}
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where μ is some unknown factor. In the classical theory $\mu = 1$

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What is the precise meaning of general covariance in the Hamiltonian formulation?

Covariance in canonical formulation

How to solve dynamics in the canonical formulation?

1) Choose a lapse function N and a shift vector N^x
2) Solve the Hamilton's equation:

$$\dot{K}_I = \{K_I, H_{\text{eff}}[N] + H_x[N^x]\},$$
$$\dot{E}^I = \{E^I, H_{\text{eff}}[N] + H_x[N^x]\};$$

3) Define the metric as: $ds^2 = -N^2 dt^2 + \frac{(E^2)^2}{E^1}(dx + N^x dt)^2 + E^1 d\Omega^2$

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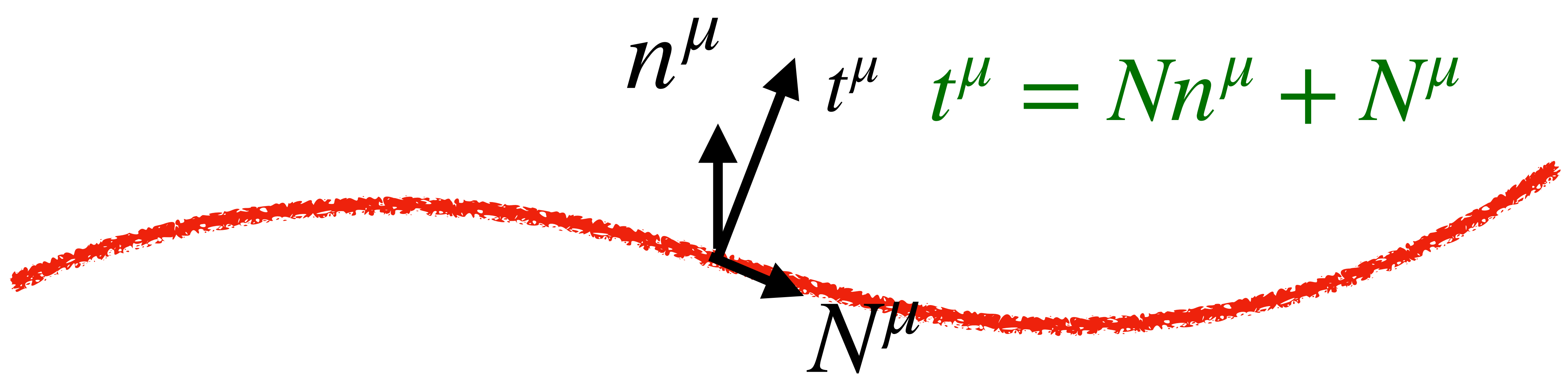
If the final metric depends on the choice of N and N^x

- **No: theory is covariant;**
- **Yes: theory is not covariant.**



General Covariance in Effective QG

$$N^x \to N^x - \epsilon[N^2 \mu E^1 (E^2)^{-2} \partial_x \alpha + (\mathscr{L}_\beta \mathfrak{N})^x]$$

$$N \to N + \epsilon[\mathscr{L}_{\alpha\mathfrak{N}+\beta} N + N \mathfrak{N}^\rho \partial_\rho \alpha], \quad \mathfrak{N} = \partial_t - N^x \partial_x$$

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By constraint algebra

$$K_I(x) \to K_I(x) + \epsilon\{K_I(x), H_{\text{eff}}[\alpha N] + H_x[\beta^x]\}$$
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$$\begin{aligned}\delta g_{\rho\sigma} \mathrm{d}x^\rho \mathrm{d}x^\sigma = {} & \mathscr{L}_{\alpha\mathfrak{N}+\beta}(g_{\rho\sigma}\mathrm{d}x^\rho \mathrm{d}x^\sigma) \\ & + \left(\frac{\Delta_1}{(E^1)^2} - \frac{2\Delta_2}{E^1 E^2}\right)(\mathrm{d}x + N^x \mathrm{d}t)^2 - \Delta_1 \mathrm{d}\Omega^2 \\ & + N^2(1-\mu)\partial_x \alpha(2\mathrm{d}x\mathrm{d}t + 2N^x(\mathrm{d}t)^2)\end{aligned}$$

$\Delta_I = 0$ **if** $H_{\rm eff}$ **is independent of** $\partial_x K_I$

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$\delta g_{\rho\sigma} \mathrm{d}x^\rho \mathrm{d}x^\sigma = \mathscr{L}_{\alpha\mathfrak{N}+\beta}(g_{\rho\sigma} \mathrm{d}x^\rho \mathrm{d}x^\sigma)$, **theory is covariant (e.g., Classical theory)**

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$$\delta g_{\rho\sigma}\mathrm{d}x^\rho\mathrm{d}x^\sigma = \mathscr{L}_{\alpha\mathfrak{N}+\beta}(g_{\rho\sigma}\mathrm{d}x^\rho\mathrm{d}x^\sigma) + \left(\frac{\Delta_1}{(E^1)^2} - \frac{2\Delta_2}{E^1E^2}\right)(\mathrm{d}x + N^x\mathrm{d}t)^2 - \Delta_1\mathrm{d}\Omega^2 + N^2(1-\mu)\partial_x\alpha(2\mathrm{d}x\mathrm{d}t + 2N^x(\mathrm{d}t)^2)$$

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Gauge trans. = spacetime diff. trans

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In general: $\delta g_{\rho\sigma}\mathrm{d}x^\rho\mathrm{d}x^\sigma \neq \mathscr{L}_{\alpha\mathfrak{N}+\beta}(g_{\rho\sigma}\mathrm{d}x^\rho\mathrm{d}x^\sigma)$.

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Two key elements: 1) the metric; 2) the constraints

General Covariance needs them to be aligned: fix metric and find $H_{\rm eff}$

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$\Delta_I = 0$ **if** $H_{\rm eff}$ **is independent of** $\partial_x K_I$

We cannot use the classical definition of $g_{\mu\nu}$

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General Covariance in Effective QG

We introduce the effective metric $g_{ab}^{(\mu)}$ defined by

$$ds^2_{(\mu)} = -N^2 dt^2 + \frac{(E^2)^2}{\mu E^1}(dx + N^x dt)^2 + E^1 d\Omega^2$$

so that $q^{xx}_{(\mu)} = \mu E^1 (E^2)^{-2}$

[see also other works by M. Bojowald's, A. Alonso-Bardaji, and D. Brizuela, and so on]

$$\begin{aligned}
\{H_x[N_1^x], H_x[N_2^x]\} &= H_x[N_1^x \partial_x N_2^x - N_2^x \partial_x N_1^x], \\
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In the classical theory, the factor here has the geometric interpretation of q^{xx}.

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Theorem 1. *Suppose the constraint algebra* (3.1). *The associated Hamiltonian theory is covariant with respect to $g^{(\mu)}_{\rho\sigma}$ given in* (3.10), *namely equation*

$$\delta g^{(\mu)}_{\rho\sigma} = \mathcal{L}_{\alpha\mathfrak{N}+\beta} g^{(\mu)}_{\rho\sigma}$$

holds for all smeared function α and smeared vector field $\beta^x\partial_x$ if and only if

(i) H_{eff} is independent of $\partial_x^n K_1$ for all $n \geq 1$;

(ii) The following equation is satisfied for all phase space independent α and N:

$$\alpha\,\{\mu S, H_{\text{eff}}[N]\} = \{\mu S, H_{\text{eff}}[\alpha N]\}\,. \tag{3.19}$$

[CZ, J. Lewandowski, Y. Ma, J. Yang, 2025]

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If $H_{\rm eff}$ exists and if it exists, it is unique?

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Derivative the covariance equation

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Aim: to get the expression of $H_{\rm eff}$

Derivative the covariance equation

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$H_{\rm eff}$ is a scalar density with weight 1 ⇒

- $H_{\rm eff} = E^2 F$
- F is a scalar field, i.e. function of elementary scalars

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Additional assumption: $\partial_x^n K_2$ is excluded as in classical theory;

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+

Covariance conditions:

(i) H_{eff} *is independent of* $\partial_x^n K_1$ *for all* $n \geq 1$*;*

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Aim: to get the expression of H_{eff}

H_{eff} is a scalar density with weight 1 $\Rightarrow$

- $H_{\text{eff}} = E^2 F$
- F is a scalar field, i.e. function of elementary scalars

Additional assumption: $\partial_x^n K_2$ is excluded as in classical theory;

F should be a function of basic scalars formed from $K_I,\ E^I,\ \partial_x E^I\ \partial_x^2 E^I$

- $\partial_x^n K_1$ is excluded due to the requirement (1);
- $\partial_x^n E^I\ (n > 2)$ are excluded, otherwise $\{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} = \cdots + N_1 \partial_x^{n-1} N_2 - N_2 \partial_x^{n-1} N_1$

Derivative the covariance equation

Constraint algebra:

$$\{H_x[N_1^x], H_x[N_2^x]\} = H_x[N_1^x \partial_x N_2^x - N_2^x \partial_x N_1^x],$$
$$\{H_{\text{eff}}[N], H_x[M^x]\} = -H_{\text{eff}}[M^x \partial_x N],$$
$$\{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} = H_x[\mu E^1 (E^2)^{-2} (N_1 \partial_x N_2 - N_2 \partial_x N_1)]\,.$$

+

Covariance conditions:

(i) H_{eff} is independent of $\partial_x^n K_1$ for all $n \geq 1$;

(ii) The following equation is satisfied for all phase space independent α and N:

$$\alpha \{\mu S, H_{\text{eff}}[N]\} = \{\mu S, H_{\text{eff}}[\alpha N]\}\,. \qquad (3.19)$$

Aim: to get the expression of H_{eff}

H_{eff} is a scalar density with weight 1 ⇒

- $H_{\text{eff}} = E^2 F$
- F is a scalar field, i.e. function of elementary scalars

Additional assumption: $\partial_x^n K_2$ is excluded as in classical theory;

F should be a function of basic scalars formed from K_I, E^I, $\partial_x E^I$ $\partial_x^2 E^I$

- $\partial_x^n K_1$ is excluded due to the requirement (1);
- $\partial_x^n E^I$ $(n > 2)$ are excluded, otherwise $\{H_{\text{eff}}[N_1], H_{\text{eff}}[N_2]\} = \cdots + N_1 \partial_x^{n-1} N_2 - N_2 \partial_x^{n-1} N_1$

The remaining basic scalars are: $s_1 = E^1, s_2 = K_2, s_3 = \dfrac{K_1}{E^2}, s_4 = \dfrac{\partial_x E^1}{E^2},\ s_5 = \dfrac{\partial_x s_4}{E^2}, s_6 = \dfrac{\partial_x E^1}{K_1}, s_7 = \dfrac{\partial_x s_4}{K_1}\,.$

Derivative the covariance equation

Constraint algebra:

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Covariance conditions:

(i) H_{eff} *is independent of* $\partial_x^n K_1$ *for all* $n \geq 1$*;*

(ii) The following equation is satisfied for all phase space independent α *and* N*:*

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s_6, s_7 is excluded because we want H_{eff} to have regular behavior in the classical regime

The remaining basic scalars are: $s_1 = E^1, s_2 = K_2, s_3 = \dfrac{K_1}{E^2}, s_4 = \dfrac{\partial_x E^1}{E^2},\ s_5 = \dfrac{\partial_x s_4}{E^2}, s_6 = \dfrac{\partial_x E^1}{K_1}, s_7 = \dfrac{\partial_x s_4}{K_1}\,.$

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(i) $H_{\rm eff}$ is independent of $\partial_x^n K_1$ for all $n \geq 1$;

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Aim: to get the expression of $H_{\rm eff}$

$H_{\rm eff}$ is a scalar density with weight 1 ⇒

- $H_{\rm eff} = E^2 F$
- F is a scalar field, i.e. function of elementary scalars

we finally get

$$H_{\rm eff} = -2E^2\left[\partial_{s_1}M_{\rm eff} + \frac{\partial_{s_2}M_{\rm eff}}{2}s_3 + \frac{\partial_{s_4}M_{\rm eff}}{s_4}s_5 + \mathcal{R}\right], \tag{4.39}$$

where $\mathcal{R}(s_1, M_{\rm eff})$ is an arbitrary function, and $M_{\rm eff}(s_1, s_2, s_4)$ satisfies the equations (4.35) and (4.38), i.e.,

$$\begin{aligned}\frac{\mu s_1 s_4}{4G^2} &= (\partial_{s_2}M_{\rm eff})\partial_{s_2}\partial_{s_4}M_{\rm eff} - (\partial_{s_4}M_{\rm eff})\partial_{s_2}^2 M_{\rm eff},\\ (\partial_{s_2}\mu)\partial_{s_4}M_{\rm eff} &- (\partial_{s_4}\mu)\partial_{s_2}M_{\rm eff} = 0.\end{aligned} \tag{4.40}$$

[CZ, J. Lewandowski, Y. Ma, J. Yang, 2025]

Solutions to the covariance equation

$$\frac{\mu s_1 s_4}{4G^2} = (\partial_{s_2} M_{\rm eff})\partial_{s_2}\partial_{s_4} M_{\rm eff} - (\partial_{s_4} M_{\rm eff})\partial^2_{s_2} M_{\rm eff},$$
$$(\partial_{s_2}\mu)\partial_{s_4} M_{\rm eff} - (\partial_{s_4}\mu)\partial_{s_2} M_{\rm eff} = 0. \tag{4.40}$$

μ is a function of s_1 and $M_{\rm eff}$

$$\rightarrow \quad \frac{\mu s_1}{8}\left[(s_4)^2 + \mathcal{Z}\right] = \frac{1}{2}(\partial_{s_2} M_{\rm eff})^2,$$ **for arbitrary functions $\mathscr{Z}$ of $s_1 \equiv E^1$ and $M_{\rm eff}$**

Solutions to the covariance equation

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Solve EOM

$$M_{\text{eff}} = M \text{ with } \partial_x M(x) + 2x\mathcal{R}(x^2, M(x)) = 0.$$

Solutions to the covariance equation

$$\frac{\mu s_1 s_4}{4G^2} = (\partial_{s_2} M_{\text{eff}})\partial_{s_2}\partial_{s_4} M_{\text{eff}} - (\partial_{s_4} M_{\text{eff}})\partial^2_{s_2} M_{\text{eff}},$$
$$(\partial_{s_2}\mu)\partial_{s_4} M_{\text{eff}} - (\partial_{s_4}\mu)\partial_{s_2} M_{\text{eff}} = 0. \quad (4.40)$$

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Solve EOM

$$M_{\text{eff}} = M \text{ with } \partial_x M(x) + 2x\mathcal{R}(x^2, M(x)) = 0.$$

$$\mathrm{d}s^2 = \frac{1}{4}N^2\mathcal{Z}\mathrm{d}t_s^2 - \frac{4}{\mu\mathcal{Z}}\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2,$$

$$N(x) = \exp\left(2\int x(\partial_{M_{\text{eff}}}\mathcal{R})(x)\mathrm{d}x\right)$$

Solutions to the covariance equation

$$\frac{\mu s_1 s_4}{4G^2} = (\partial_{s_2} M_{\rm eff})\partial_{s_2}\partial_{s_4} M_{\rm eff} - (\partial_{s_4} M_{\rm eff})\partial_{s_2}^2 M_{\rm eff},$$
$$(\partial_{s_2}\mu)\partial_{s_4} M_{\rm eff} - (\partial_{s_4}\mu)\partial_{s_2} M_{\rm eff} = 0. \quad (4.40)$$

μ is a function of s_1 and $M_{\rm eff}$

$$\frac{\mu s_1}{8}\left[(s_4)^2 + \mathcal{Z}\right] = \frac{1}{2}(\partial_{s_2} M_{\rm eff})^2,$$ **for arbitrary functions $\mathscr{Z}$ of $s_1 \equiv E^1$ and $M_{\rm eff}$**

Solve EOM

$$M_{\rm eff} = M \text{ with } \partial_x M(x) + 2x\mathcal{R}(x^2, M(x)) = 0.$$

$$ds^2 = \frac{1}{4}N^2\mathcal{Z}dt_s^2 - \frac{4}{\mu\mathcal{Z}}dx^2 + x^2 d\Omega^2,$$

$$N(x) = \exp\left(2\int x(\partial_{M_{\rm eff}}\mathcal{R})(x)dx\right)$$

[CZ, Z. Cao, 2025]

Given $$ds^2 = -F(x;m)dt_s^2 + H(x;m)^{-1}dx^2 + x^2 d\Omega^2,$$ $\longrightarrow$ $$F = -\frac{1}{4}N^2\mathcal{Z}, \quad H = -\frac{1}{4}\mu\mathcal{Z}.$$

LQG motivated metrics: $\bar{\mu}$-scheme LQBH

$$s_1 = E^1, s_2 = K_2, s_3 = \frac{K_1}{E^2}, s_4 = \frac{\partial_x E^1}{E^2}, s_5 = \frac{\partial_x s_4}{E^2}.$$

$$\mathrm{d}s^2_{(1)} = -f_1 \mathrm{d}t^2 + f_1^{-1}\mathrm{d}x^2 + x^2 \mathrm{d}\Omega^2,$$

$$f_1 = 1 - \frac{2M}{x} + \frac{\zeta^2}{x^2}\left(1 - \frac{2M}{x}\right)^2,$$

$$M^{(1)}_{\mathrm{eff}} = \frac{\sqrt{s_1}}{2} + \frac{\sqrt{s_1}^3 \sin^2\left(\frac{\zeta s_2}{\sqrt{s_1}}\right)}{2\zeta^2} - \frac{\sqrt{s_1}(s_4)^2}{8} e^{\frac{2i\zeta s_2}{\sqrt{s_1}}},$$

[CZ, J. Lewandowski, Y. Ma, J. Yang, 2025]

OR

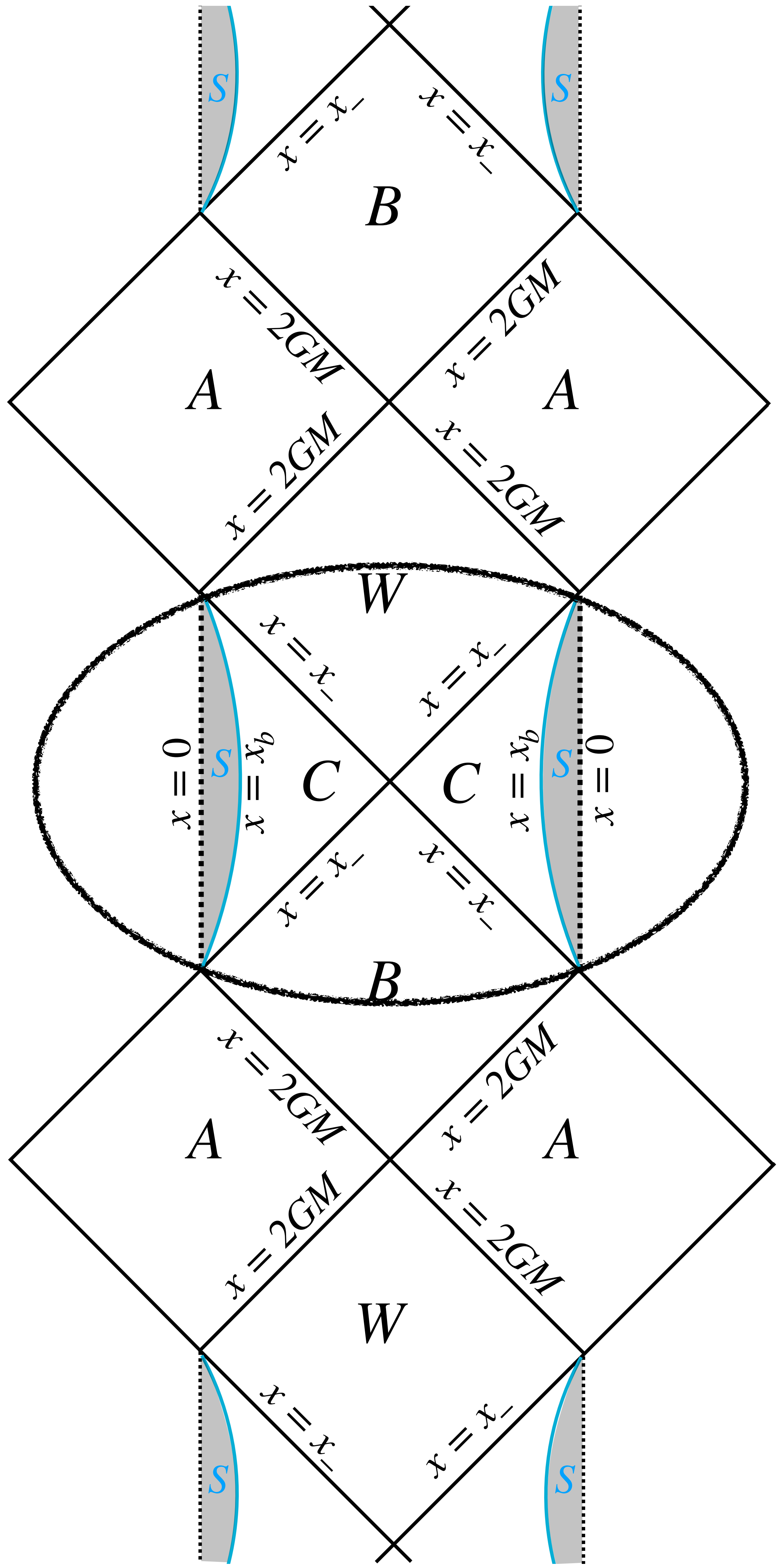
$$f_1 = 1 - \frac{2M}{x} + \frac{4\zeta^2 M^2}{x^4}$$

$$M_{\mathrm{eff}} = \frac{\sqrt{s_1}^3}{2\zeta^2}\sin^2\left(\frac{\zeta s_2}{\sqrt{s_1}} \pm \frac{2\zeta\Xi}{s_1}\right) \mp \frac{s_1\sqrt{(s_4)^2 - 4}\,\sin\left(\frac{2\zeta s_2}{\sqrt{s_1}} \pm \frac{4\zeta\Xi}{s_1}\right)}{4\zeta}.$$

for arbitrary $\Xi(s_1, s_4)$

[CZ, Z. Cao, 2025]

Transition region connecting BH and WH



LQG motivated metrics: $\bar{\mu}$-scheme LQBH

$$ds^2_{(2)} = -f_2 dt^2 + \mu_2^{-1} f_2^{-1} dx^2 + x^2 d\Omega^2,$$

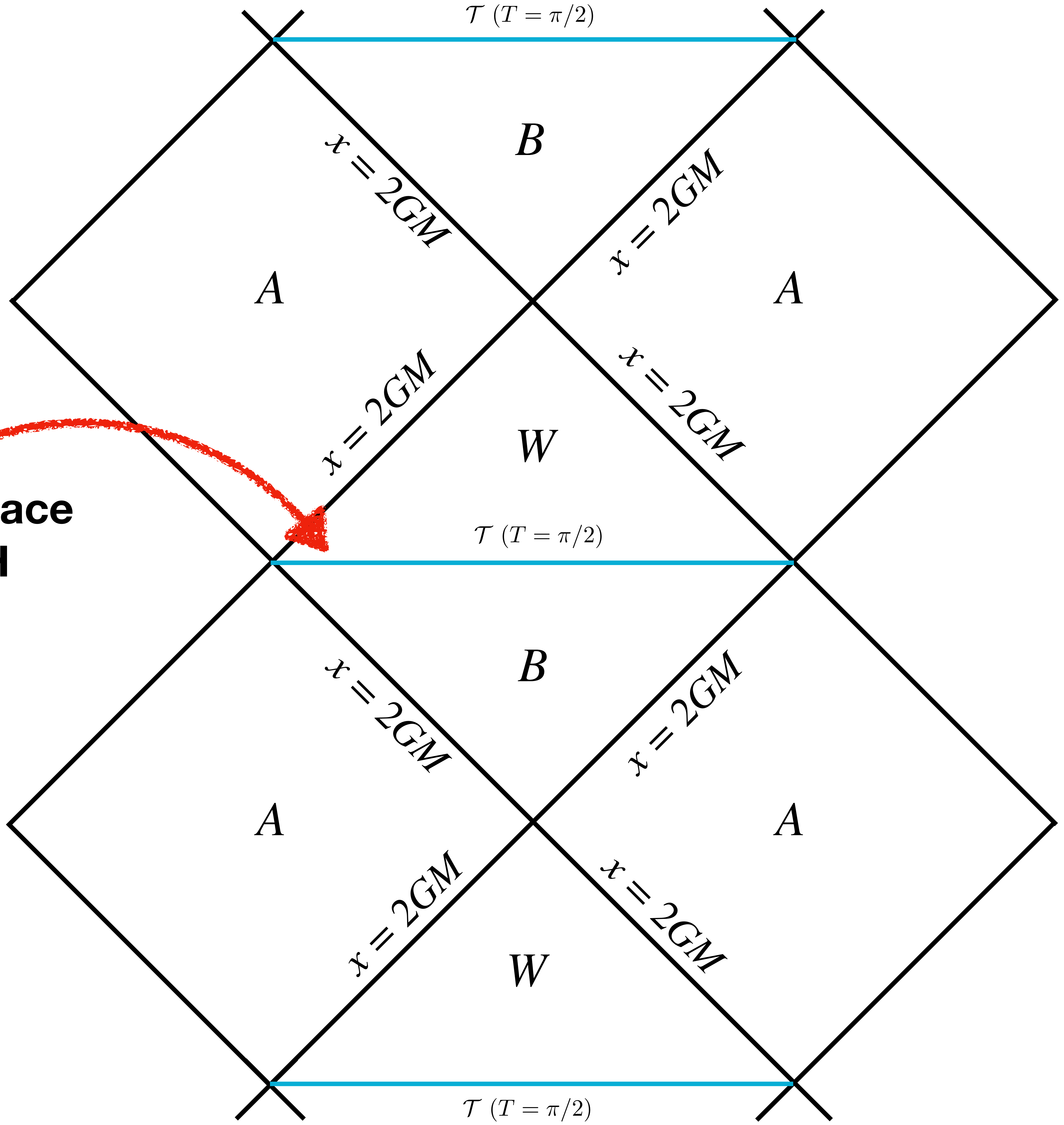
$$f_2 = 1 - \frac{2M}{x}, \qquad \mu_2 = 1 + \frac{\zeta^2}{x^2}\left(1 - \frac{2M}{x}\right).$$

$$M_{\text{eff}}^{(2)} = \frac{\sqrt{s_1}}{2} + \frac{\sqrt{s_1}^3 \sin^2\left(\frac{\zeta s_2}{\sqrt{s_1}}\right)}{2\zeta^2} - \frac{\sqrt{s_1}(s_4)^2 \cos^2\left(\frac{\zeta s_2}{\sqrt{s_1}}\right)}{8}$$

[CZ, J. Lewandowski, Y. Ma, J. Yang, 2025]

$$s_1 = E^1, s_2 = K_2, s_3 = \frac{K_1}{E^2}, s_4 = \frac{\partial_x E^1}{E^2}, s_5 = \frac{\partial_x s_4}{E^2}.$$

Transition surface connecting BH and WH



A metric without Cauchy horizon

$$\mathrm{d}s^2_{(3)} = -\bar{f}_3^{(n)}\mathrm{d}t^2 + \bar{\mu}_3^{-1}\left(\bar{f}_3^{(n)}\right)^{-1}\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2,$$

$$\bar{f}_3^{(n)}(x) = 1 - (-1)^n\frac{x^2}{\zeta^2}\arcsin\left(\frac{2GM\zeta^2}{x^3}\right) - \frac{n\pi x^2}{\zeta^2},$$

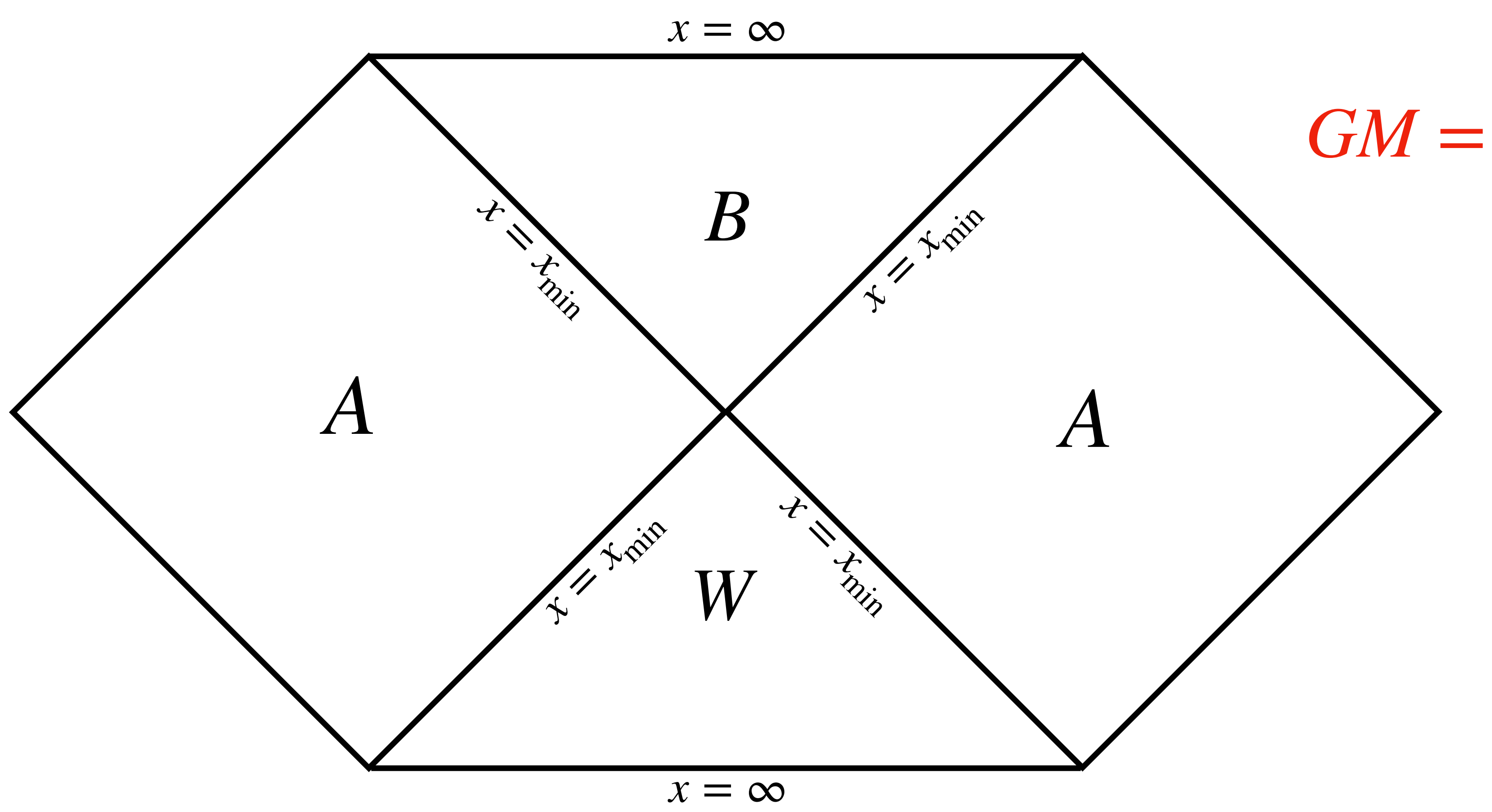
$$\bar{\mu}_3(x) = 1 - \frac{4G^2\zeta^4M^2}{x^6}.$$

$$M_{\rm eff}^{(3)} = \frac{\sqrt{s_1}^3}{2G\zeta^2}\sin\left(\frac{\zeta^2}{s_1}\left[1+(s_2)^2-\frac{(s_4)^2}{4}\right]\right)$$

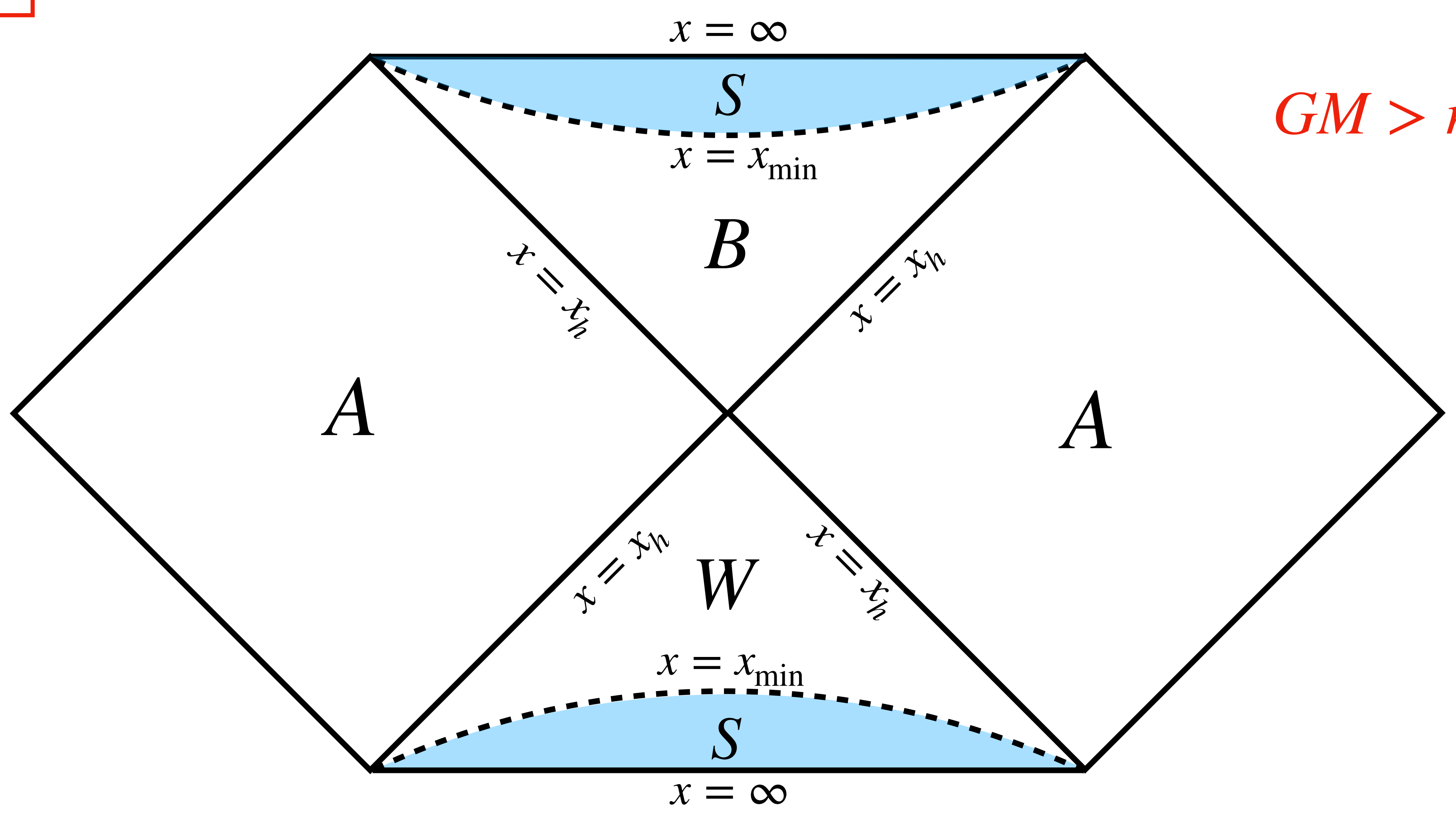
[CZ, J. Lewandowski, Y. Ma, J. Yang, 2025]

$$m_o = \frac{\zeta}{2}\left(\frac{2}{\pi}\right)^{3/2}.$$

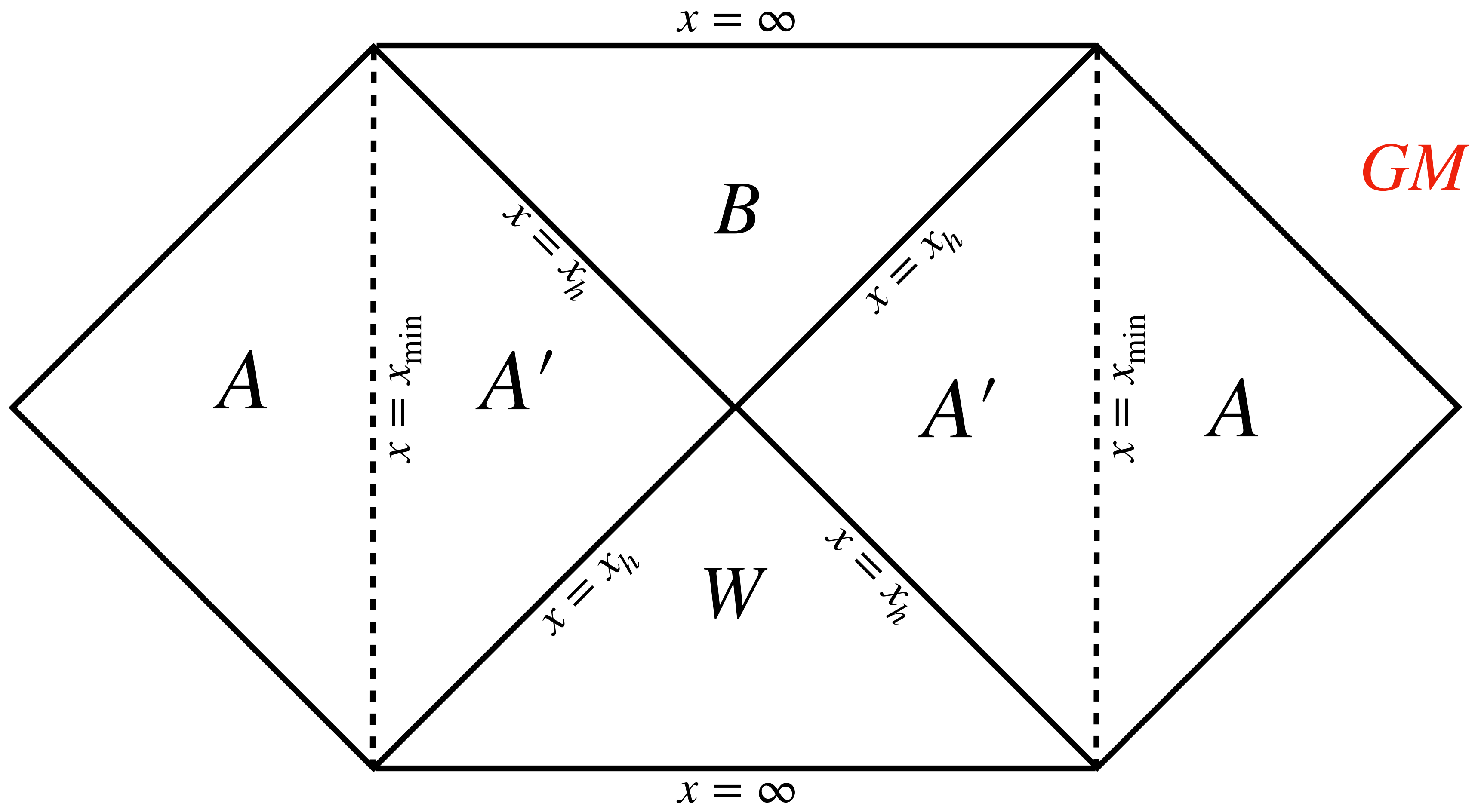
$$\left(2GM\zeta^2\right)^{1/3} \equiv x_{\min}.$$



$GM = m_o$



$GM > m_o$



$GM < m_o$

Hayward metric

$$ds^2 = -F_H\,dt^2 + (F_H)^{-1}\,dx^2 + x^2 d\Omega^2$$

$$F_H = 1 - \frac{2mx^2}{x^3 + 2\zeta^2 m}.$$

$$s_1 = E^1,\ s_2 = K_2,\ s_3 = \frac{K_1}{E^2},\ s_4 = \frac{\partial_x E^1}{E^2},\ s_5 = \frac{\partial_x s_4}{E^2}.$$

$$-\frac{\left(2\zeta^2 M_{\text{eff}} + \sqrt{s_1}^3\right)\sqrt{(s_4)^2 - 4F_H(M_{\text{eff}}, \sqrt{s_1})}}{8s_1 + 2\zeta^2\left((s_4)^2 - 4\right)}$$

$$= \frac{2\sqrt{s_1}^5 \operatorname{arctanh}\left(\frac{\zeta\sqrt{(s_4)^2 - 4F_H(M_{\text{eff}}, \sqrt{s_1})}}{\sqrt{4s_1 + \zeta^2((s_4)^2 - 4)}}\right)}{\zeta\sqrt{4s_1 + \zeta^2\left((s_4)^2 - 4\right)}^3} \mp \frac{1}{2}\sqrt{s_1}s_2,$$

[CZ, Z. Cao, 2025]

Matter coupling: coupled to EM field

Classical theory:

$$A_\rho \mathrm{d}x^\rho = \Phi \mathrm{d}t + \Gamma \mathrm{d}x,$$

$$\Phi = \Psi + \left(N \frac{\sqrt{E^1}}{E^2} + N^x \right) \Gamma.$$

Covariant effective theory:

$$A_\rho^{(\mu)} \mathrm{d}x^\rho = \left[\Psi + \left(N \frac{\sqrt{\mu E^1}}{E^2} + N^x \right) \Gamma \right] \mathrm{d}t + \Gamma \mathrm{d}x,$$

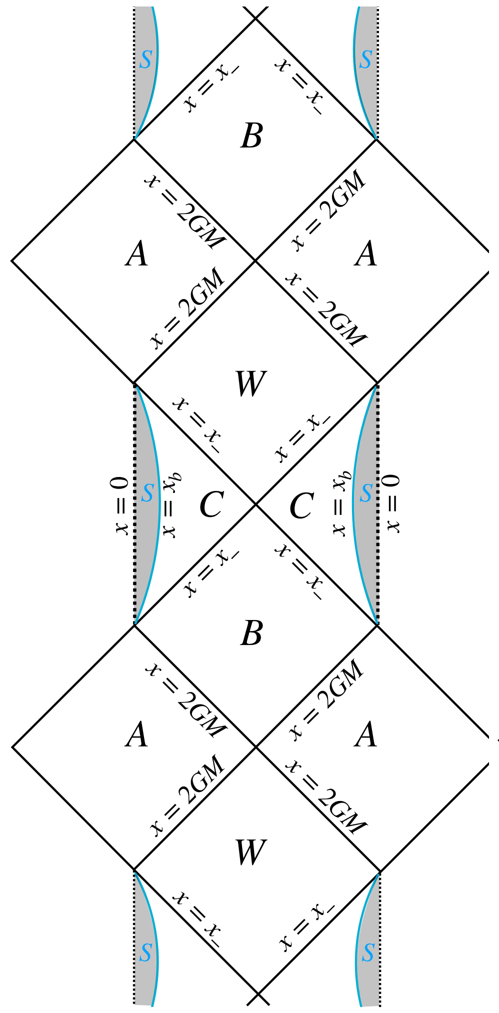
[J. Yang, Y. Ma, CZ, 2025]

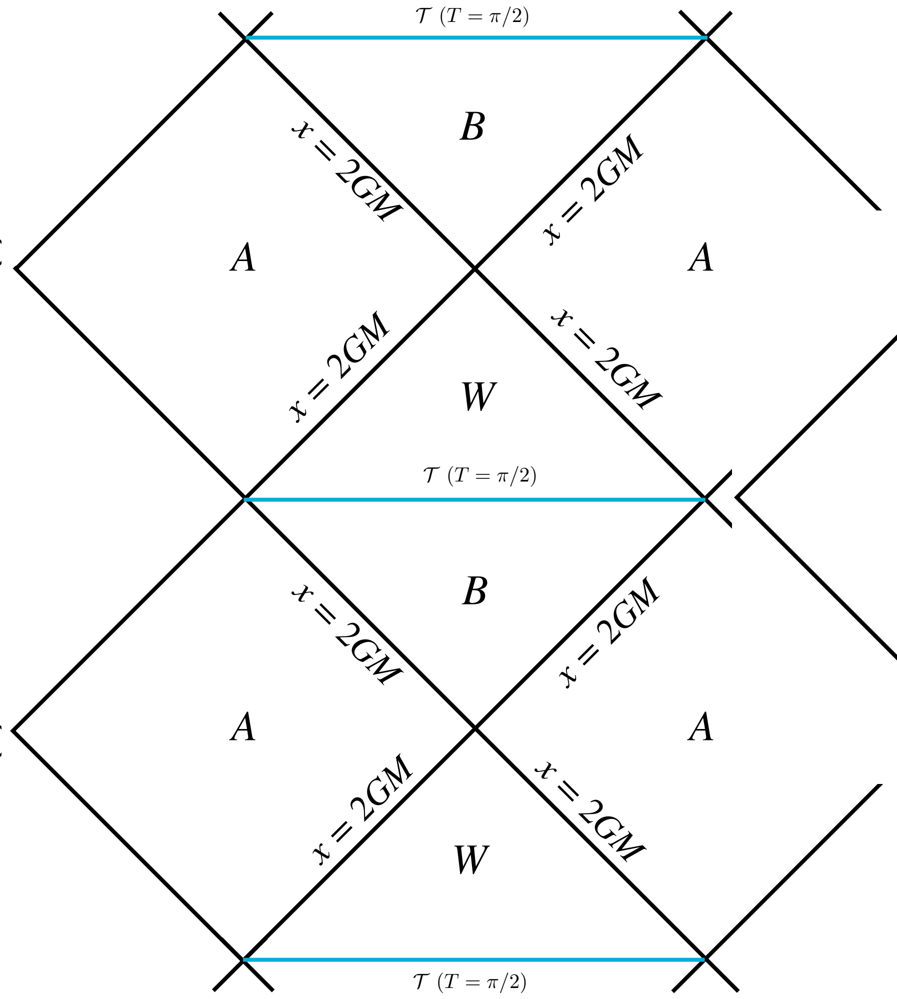
Conclusion and outlook

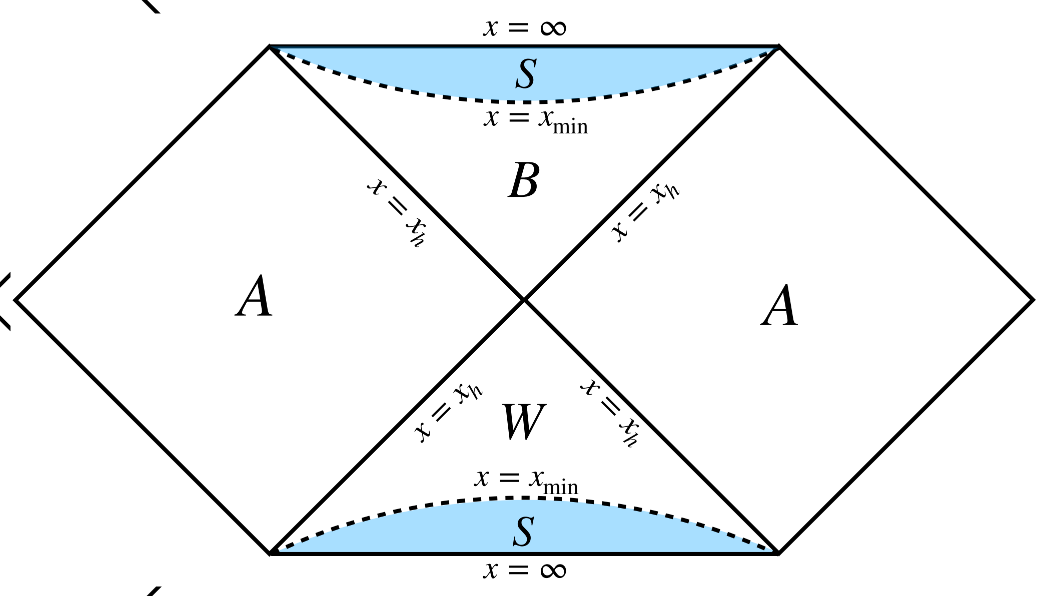
- **The sufficient and necessary condition for covariance;**

(i) $H_{\rm eff}$ **is independent of derivatives of** K_1**;**
(ii) $\{S(x), H_{\rm eff}[\alpha N]\} = \alpha(x)\{S(x), H_{\rm eff}[N]\}$ **for any phase space independent functions** α **and** N**.**

- **Three solutions to the covariance equation:**







- **Covariance equation for the effective Hamiltonian constraint;**

$$H_{\rm eff} = -2E^2\left[\partial_{s_1} M_{\rm eff} + \frac{\partial_{s_2} M_{\rm eff}}{2} s_3 + \frac{\partial_{s_4} M_{\rm eff}}{s_4} s_5 + \mathcal{R}\right], \quad (6)$$

where $\mathcal{R}$ is an arbitrary function of s_1 and $M_{\rm eff}$, and $M_{\rm eff}$ depending on s_1, s_2, s_4 is a solution to:

$$\frac{\mu s_1 s_4}{4} = (\partial_{s_2} M_{\rm eff})\partial_{s_2}\partial_{s_4} M_{\rm eff} - (\partial_{s_4} M_{\rm eff})\partial^2_{s_2} M_{\rm eff}, \quad (7a)$$

$$(\partial_{s_2}\mu)\partial_{s_4} M_{\rm eff} - (\partial_{s_2} M_{\rm eff})\partial_{s_4}\mu = 0. \quad (7b)$$

- **Reconstruction of dynamics from geometry**

$$\mathrm{d}s^2 = \frac{1}{4} N^2 \mathcal{Z} \mathrm{d}t_s^2 - \frac{4}{\mu \mathcal{Z}} \mathrm{d}x^2 + x^2 \mathrm{d}\Omega^2,$$

Thanks for your attention!