Hilbert Space Non-Seperability as a Virtue

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Non-seperable Hilbert spaces are large and unwieldy objects ill suited to most physical situations.

Nevertheless, there are situations where this largeness is a virtue.

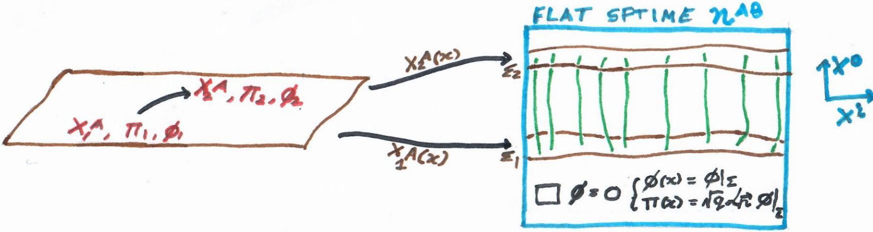
I will describe two examples in the context of free quantum fields on flat spacetime where non-seperability helps alleviate problems.

1. In the first example quantum states at different times turn out to live in *unitarily-inequivalent* (separable) Hilbert spaces. In this context, we shall see that non-separability enables a useful articulation of unitary evolution between such states.
2. In the second, non-seperability removes the intuitive tension between Lorentz transformations and spacetime discreteness by enabling a unitary representation of Lorentz transformations in the context of quantum fields living on *discrete spacetime lattices*.

Both examples involve free scalar field theory on flat spacetime re-expressed in a diffeomorphism invariant formulation known as Parameterized Field Theory.

Example 1: PFT and standard Fock Repn

- On $n{+}1$ d flat sptime with inertial coordinates X^A
  $S_0[\phi] = -\frac{1}{2}\int d^{n+1}X \eta^{AB}\partial_A\phi\partial_B\phi$.
- 'Parameterize': $X^A = X^A(x^0, x^1, .., x^n)$:
  $S_0[\phi] = -\frac{1}{2}\int d^{n+1}x\sqrt{\det\eta}\,\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$,
  $\eta_{\alpha\beta} = \eta_{AB}\partial_\alpha X^A\partial_\beta X^B$
- Consider above action also as fnal of 'new' scalar fields $X^A(x)$:
  $S[X^A, \phi] = -\frac{1}{2}\int d^{n+1}x\sqrt{\det\eta(X)}\,\eta^{\alpha\beta}(X)\partial_\alpha\phi\partial_\beta\phi$,
  Varying X^A, ϕ in S yields eqns of motion equivalent to those from varying ϕ in S_0. *S is generally covariant*
- $n+1$ decomposition: phase space (X^A, P_A, ϕ, π), constraints $C_A = P_A + h_A(X, \phi, \pi)$, Lagrange multipliers N^A ("lapse-shift").
- Phase sp variables are fns on n dim space with coordinates $(x^1, .., x^n)$. Focus on $X^A(x^1..x^n)$.
- $X^A(x^1..x^n)$ define $n{+}1$ inertial coordinates at every point p in n dimnal space i.e. a point P in flat sptime for every p. Hence, an embedding of an n-dim manifold as a spatial slice of $n{+}1$ d flat sptime.


X₁^A, π₁, φ₁
X₂^A, π₂, φ₂
X₂^A(x)
X₁^A(x)
Σ₂
Σ₁
FLAT SPTIME η^AB
X^0
X^i
□φ = 0
φ(x) = φ|Σ

Quantization

- Quantum constraints: $\hat{C}_A\Psi = 0 = (\hat{P}_A + h_A(\hat{X}, \hat{\pi}, \hat{\phi})\Psi = 0$
- Represent $\hat{X}^A$ by multiplcation, $\hat{P}^A(x) := \frac{1}{i}\frac{\delta}{\delta X^a(x)}$.

  $\hat{C}_A\Psi = 0 \Rightarrow \left(\frac{1}{i}\frac{\delta}{\delta X^A} + \hat{h}_A\right)\Psi(X, matter) = 0$

  Functional Schrodinger Eqn
- The standard Fock quantization is in H'berg picture:
  We know $\hat{\phi}$ everywhere on sptime in terms of standard Fock space annhilation, creation operators. Hence, we know operators $\hat{\phi}_X, \hat{\pi}_X$ on any slice X. Fix an initial slice X_0^A.
  If operator evoltn X_0 to arbitrary X is unitarily implemented:

  $$\hat{U}_X \;\; \hat{\phi}_{X_0} \;\; \hat{U}_X^\dagger = \hat{\phi}_X \qquad \hat{U}_X \;\; \hat{\pi}_{X_0} \;\; \hat{U}_X^\dagger = \hat{\pi}_X$$
- Then fnal Schrod pic is (X- dependent) inverse unitary image of H'berg pic. Slice dep inverse unitary image of *slice-indep* H'berg state then satisfies fnal Schro eqn. This is what happens in 1+1d.
- For higher dimn evoltn not unitarily implemented (generically), no fnal Schro pic, no slice dep states in Fock space. Dirac quantzn seems to fail in simplest model field theory...bad news for canonical QG?

Algebraic states

- Fix initial slice $X^0 = 0, X^i(x) = x^i$ with matter phase space data (ϕ, π). Consider standard Weyl algebra generated by operator correspondents of elements $W(\alpha, \beta) = e^{i\int \alpha(x)\pi(x) - \beta(x)\phi(x)}$
  Linear canonical transformations leave algebraic structure invariant. Classical evolution to any slice $X^A(x)$ is linear canonical transf. Defines slice dep automorphism $\mathcal{A}_X$ of Weyl algebra.
- Fock repn provides a repn of Weyl algebra. Fock vacuum $|0\rangle$ defines PLF ψ_0 via vacuum expectation value of algebra elements.
- For any element $\hat{W}$ of algebra can define a new PLF ψ_X via pull back of the PLF ψ_0 by $\mathcal{A}_X$:
  $\psi_X(\hat{W}) = \psi_0(\mathcal{A}_X(\hat{W}))$
- Every such PLF ψ_X defines a GNS Hilbert space $\mathcal{H}_X$ and a state $|0, X\rangle \in \mathcal{H}_X$ s.t. the exp value of algebra elements in this state reproduces the PLF. For 1+1 d these Hilbert spaces are unitary images of Fock space and the states are the Schrodinger states. For higher dimns these Hilbert space repns are unitarily inequivalent to Fock repn for generic slices X and as we saw we have an obstruction to Dirac quantization of PFT.

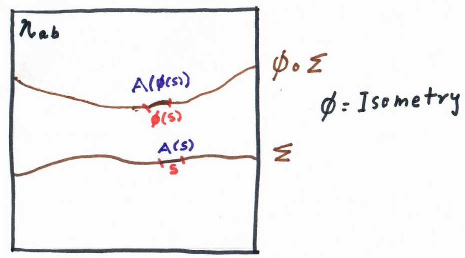
Non-seperability to the rescue

- Define ‘polymer’ repn: $\hat{X}^A(x)|F\rangle = F^A(x)|F\rangle$ with $F^A(x)$ smooth, $\langle F_1|F_2\rangle = \delta_{F_1,F_2}$. So in this repn $\hat{P}_A$ is not well defined but its exponential is.
- Define the *non-seperable* Hilb space $\mathcal{H}_{kin} = \bigoplus_F |F\rangle \otimes \mathcal{H}_F$
  Sum is over all F^A which define slices in flat sptime. Natural inner product is induced from that on polymer Hilb space and on $\mathcal{H}_F$.
- Can be shown that on $\mathcal{H}_{kin}$ suitable generalizations of finite transformations generated by constraints act *unitarily*.
- Physical states can be constructed by averaging over these unitary transformations. Group averaging inner product space of physical states yields Hilbert space $\mathcal{H}_{phys}$.
- Classical Dirac observables corresponding to standard creation, annihilation modes exist.
- Their quantum repn on $\mathcal{H}_{phys}$ can be shown to be unitarily equiv to standard Fock space repn!
- Could there be situations in qft in cs where non-seperability might help articulate a useful notion of unitarity?

Example 2: LQG type quantization of 1+1 PFT

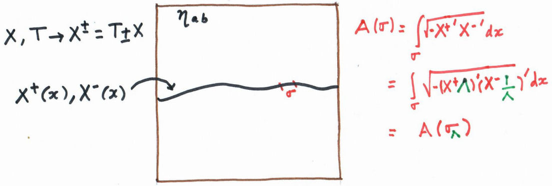
- In 2d, $\Box\phi = 0 \Rightarrow \phi(X^A) = \phi_+(X^+) + \phi_-(X^-)$ where $X^{\pm} = \frac{X^0 \pm X^1}{2}$ are light cone coordinates.
- In phase space, define 'light cone' combinations from (X^A, P_A, ϕ, π): $X^{\pm}(x) = \frac{X^0(x) \pm X^1(x)}{2}$ $P_{\pm} = \frac{P_0 \pm P_1}{2}$ and $Y_{\pm} = \frac{\pi \pm \phi'}{2}$.
  $Y_{\pm}$ are proportional to $\frac{\partial \phi}{\partial X^{\pm}}$ on shell and capture left and right moving matter modes.
- Also define suitable light cone combinations of constraints C_+, C_- which depend only on $+, -$ phase sp variables. Turns out that evolution generated by C_+ corresponds to action of diffeos d_+ on the $+$ variables, similarly evolution by C_- to diffeos d_- on $-$ variables.
- Classical generators of Lorentz transf exist and commute with constraints.

- As we shall see in LQG type quantization, kinematic states admit the interpretation of quantum matter on discrete cauchy slices, physical states admit interpretation of quantum matter on discrete sptime lattice.
  We are interested in unitary implementation of Lorentz transf on these discrete structures
- In prtclr, an 'Area operator' (equiv to length in 1+1 d) exists with discrete spectrum. Hence from LQG point of view, despite key differences between LQG and this simple model, it is also interesting to examine intuitive tension between boosts and area discreteness in this model.
- Hence we briefly digress with some general remarks on Lorentz transformations of classical area in this model (as well as on local lorentz transformations of area in the context of canonical General Relativity).



Using properties of diffeos and isometries, can see that $A(\phi(S)) = A(S)$. "Area is Lorentz Invariant"
(Remark: Area of small enough surface for arbitrary g_{ab} is LLI!)

Explicitly, in 1+1 PFT:



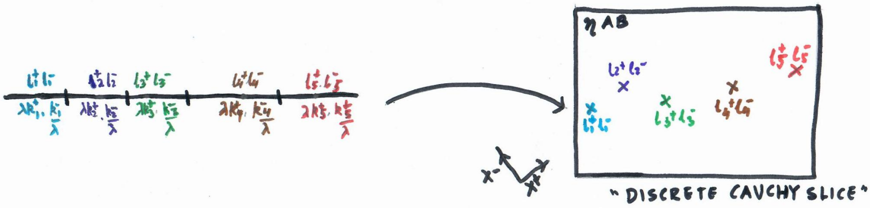
Quantum States:

In quantum theory $X^{\pm}(x)$ become operators. In LQG type repn they have discrete spectra: Fix a real, positive number λ , then $\hat{X}^{+}(x) \to \lambda \mathbf{Z}$ and $\hat{X}^{-}(x) \to \mathbf{Z}/\lambda$

Area Spectrum indep of choice of λ, spectrum is discrete.

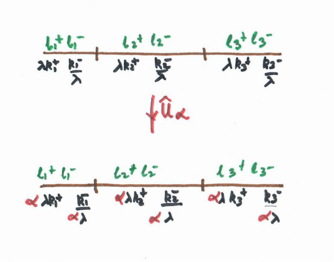
Left and Right moving scalar field degrees of freedom admit suitable LQG type repn with integer valued quantum numbers.

The kinematic Hilbert space $\mathcal{H}^{\lambda}_{kin}$ is spanned by uncountable basis of "charge network' states on 1d graphs with edges labelled by embedding charges and matter charges:



Action of boosts on Charge Network States

- Classically, finite boosts act only on the embedding sector through:
  $(X^+(x), P_+(x)) \rightarrow (\alpha X^+(x), \frac{1}{\alpha} P_+(x)),$
  $(X^-(x), P_-(x)) \rightarrow (\frac{1}{\alpha} X^-(x), \alpha P_-(x)).$
  This action commutes with that of the constraints.
- Quantum mechanically, these transformations are not defined on fixed λ sector. Hence we introduce an even larger degree of non-seperability by summing over all these sectors:
  $\mathcal{H}_{kin} := \oplus_{\alpha>0} \mathcal{H}^{\alpha}_{kin}$.
- Action of boost is then:



This action of boosts can be shown to be unitary. Because of explicitly boost invariant expression of classical area, turns out that in quantum theory Area operator is *boost invariant*:
$\hat{U}_\alpha \hat{A}(\sigma) \hat{U}_\alpha^\dagger = \hat{A}(\sigma)$
Thus, the larger degree of nonseperability, due to sum over fixed λ sectors, resolves tension between discrete areas and boost inv.

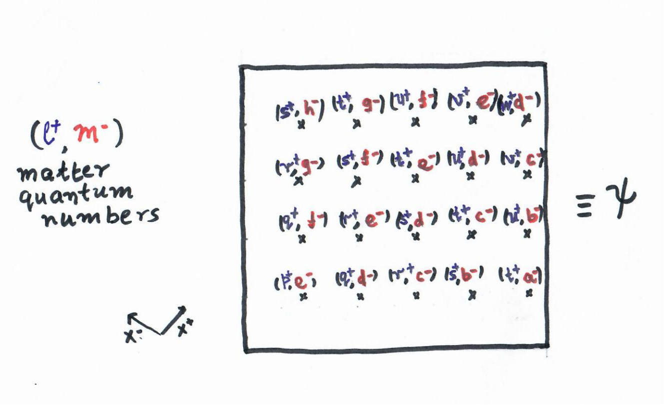
What about Physical States ?

$C_\pm$ generate diffeos $d_\pm$ acting on $\pm$ fields.

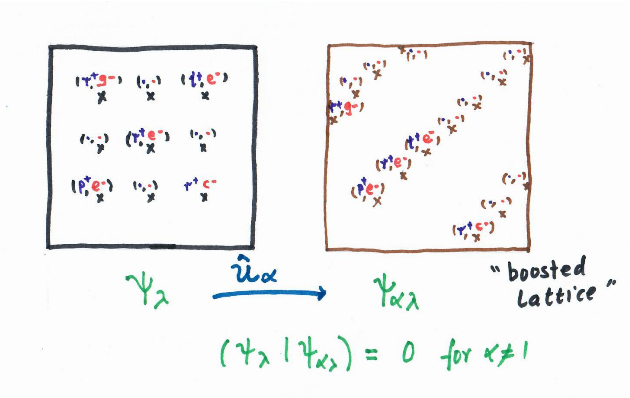
$d_\pm$ act unitarily on charge nets similar to diffeo action on spin nets in LQG. Physical state space can then be constructed via Group Averaging over (unitary) action of all d_+, d_-.

A physical state is then a sum over kinematic charge net states. Each summand defines discrete Cauchy slice with quantum matter. Because the $\pm$ diffeos are represented in anomaly free manner, all these different slices with quantum matter fit *consistently* into a single spacetime lattice with quantum matter.

Thus, Physical States admit the interpretation of Quantum Matter on a light cone lattice with spacing $\lambda, 1/\lambda$ along the $+, -$ null directions:



Lorentz transformations map physical state ψ_λ on one lattice to $\psi_{\lambda'}$ on a ‘boosted lattice’



Grp Averaging does not change the value of λ. Each fixed λ sector is *seperable*. Tension between spacetime discreteness and LI is resolved by non-seperability due to admission of all λ sectors.

Some Remarks:

- Since each fixed λ sector lives on a single sptime lattice and different λ sectors are related by unitary transformations, we could take a view that each fixed λ sector corresponds to physics seen by fixed observer and unitary transf just correspond to same physics seen by different boosted observers. From this point of view, physical state space non-seperability is related to observer perspectives.
- Recall that in this model area is classically and qmly LI. In the case of gravity and LLI, there are too many complicated issues in the quantum theory and it is difficult to see how the LLI of area of small surfaces could emerge.
- Note that the beautiful work of Carlo and Simone focuses on Lorentz contraction, hence on surfaces which are defined by different simultaneity cuts of sptime world sheet of physical object -these surfaces are not boosted images of each other.

In Conclusion

While Non-Seperable Hilbert spaces might alleviate some problems, ALL problems are alleviated by Non-Seperable Friends: