Fermions on a LQC background

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Prof. Jerzy Lewandowski

*This work is dedicated to the memory of **Jurek Lewandowski** a brilliant scientist, great mentor and collaborator, and a wonderful friend.*

Plan of the talk

1. Motivation & Context
2. Classical Setup: Fermions on the Closed FLRW
3. Closed FLRW in LQC: Key features
4. Fermions on Quantum Geometry
5. Physical Interpretation of Modified Background
6. Conclusion & Outlook

Why study fermions in LQC?

- Most work in LQC uses **scalar fields** as clocks or matter sources; the big bang is replaced by a robust, universal **quantum bounce**.
- **Perturbations** sourced by **bosonic** fields—such as scalar, vector, and tensor fields—have been extensively studied in the context of LQC.
- **Fermions**—fundamental to the Standard Model—remain largely unexplored in LQC [apart from pioneering studies by G. Mena-Marugán *et al.*].
- Could fermions produce **distinct backreaction** effects or **new quantum-gravitational phenomena**?

Key questions

- How do fermionic fields influence the quantum bounce?
- Can they induce a **mode-dependent "rainbow" geometry**?
- What is the role of fermionic perturbations in early-universe quantum gravity?

Spacetime geometry:

- Manifold: $\mathcal{M} = \mathbb{S}^3 \times \mathbb{R}$
- Background metric:

$$ds^2 = -N_0^2(x^0)(dx^0)^2 + a^2(x^0)[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)].$$

- Temporal coordinate $x^0 \in \mathbb{R}$, spatial $\chi, \theta, \phi \in \mathbb{S}^3$, spatial curvature: $k = +1$.

Matter content (fermion):

- Two-component Weyl spinors $\phi_A, \chi_{A'}$.
- Mass m, coupled via Dirac action in curved spacetime.

Dirac action (from tetrad formalism):

$$I_{\text{ferm}} = -\frac{i}{2}\int d^4x\, e\left(\bar{\phi}^{A'} e^{\mu}_{AA'} D_\mu \phi^A + \text{H.c.}\right) - \frac{m}{\sqrt{2}}\int d^4x\, e\left(\chi_A \phi^A + \text{H.c.}\right)$$

Goal: Quantize fermions on this background; study backreaction on geometry.

- **Mode expansion on $\mathbb{S}^3$** [D'Eath and Halliwell, 1987]:

Use **spinor harmonics** $\rho_A^{nq}, \bar{\sigma}_A^{nq}$ as eigenmodes of 3D Dirac operator:

$$-in_{AA'}e^{BA'j(3)}D_j\rho_B^{nq} = +\lambda_n\rho_A^{nq}, \qquad \lambda_n = n + \tfrac{3}{2}, \qquad (n = 0, 1, 2, \ldots)$$

Each λ_n has a degeneracy $d_n = (n+1)(n+2)$, reflecting the symmetry of $\mathbb{S}^3$. This corresponds to d_n linearly independent spinor harmonics, with the label $q = 1, \ldots, d_n$ distinguishing the degenerate states at energy level n.

This enables decomposition into **discrete fermionic modes** (n, p).

- **After expansion:** Each mode (n, p) described by complex functions of x^0:

$$m_{np}(x^0), \quad s_{np}(x^0), \quad t_{np}(x^0), \quad r_{np}(x^0).$$

- **Effective action per mode:** With $(x, y) \equiv$ **Grassmann variables** associated with pairs (m_{np}, s_{np}) or (t_{np}, r_{np}):

$$I_{np} = \int dt\, N_0 \left[\frac{i}{2N_0}(x\dot{\bar{x}} + \bar{x}\dot{x} + y\dot{\bar{y}} + \bar{y}\dot{y}) + \frac{\lambda_n}{a}(\bar{x}x + \bar{y}y) - m(yx + \bar{x}\bar{y})\right].$$

- **Hamiltonian:** Following fundamental Dirac bracket $[x, \bar{x}]^* = -i$ and $[y, \bar{y}]^* = -i$,

$$H_{np} = \lambda_n a^{-1}(x\bar{x} + y\bar{y}) + m(yx + xy).$$

Key: Coupling between fermion modes (x, y) and scale factor $a(t)$.

Fermionic wave function

- Fermion Hamiltonian $\to$ operator, by nonvanishing canonical **anticommutation** relations: $\{\hat{x}, \hat{\bar{x}}\} = 1$, and $\{\hat{y}, \hat{\bar{y}}\} = 1$.
- **Holomorphic representation** [Berezin, 1966]: $\hat{\bar{x}} = \partial/\partial x$, $\hat{\bar{y}} = \partial/\partial y$.
- For each (n, p), ψ depends only on the unbarred variables: $\psi(x, y) \equiv \psi(m, s, r, t)$.
- The dynamics are governed by the **Schrödinger equation**:

$$
\begin{aligned}
i\hbar\partial_{x^0}\psi(x^0; x, y) &= \sum_{np} \hat{H}_{np}(x, y)\psi(x^0; x, y) \\
&= N_0 \sum_{np} \left[\lambda_n(\widehat{x\bar{x}} + \widehat{y\bar{y}}) + m(\widehat{yx} + \widehat{\bar{x}\bar{y}})\right] \psi(x^0; x, y).
\end{aligned}
$$

- Due to mode decoupling, the total wave function factorizes:

$$
\psi = \prod_{np} \psi_{np}, \quad \text{with each } \psi_{np} \text{ evolving independently under } \hat{H}^{(\tau)}_{np}.
$$

Harmonic time gauge: Lapse $N_\tau = a^3 \to$ simplifies matter-geometry coupling; each mode Hamiltonian becomes

$$
\hat{H}^{(\tau)}_{np} = \lambda_n \ell^{-2} V^{2/3}(\widehat{x\bar{x}} + \widehat{y\bar{y}}) + m\ell^{-3} V(\widehat{yx} + \widehat{\bar{x}\bar{y}}), \qquad V = \ell^3 a^3.
$$

- Physical Hilbert space $\mathscr{H}_{np}$ for each (n, p) is spanned by **orthonormal basis** being solutions to the **stationary state** equation $\hat{H}^{(\tau)}_{np}\psi_{np} = E_{np}\psi_{np}$):

$$\left[\frac{\lambda_n}{\ell^2}V^{2/3}\left(-1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{m}{\ell^3}V\left(yx + \frac{\partial^2}{\partial x \partial y}\right)\right]\psi_{np} = E_{np}\psi_{np}.$$

- Only four orthonormal solutions:

Eigenenergy	Eigenfunction	Status
$E^{(0)}_{np} = -w_n$	$\psi^{(0)}_{np} = \mathrm{N}^{(0)}_n\left(1 + \frac{m\ell^{-3}V}{\lambda_n \ell^{-2}V^{2/3} + w_n}xy\right)$	Pair ground state
$E^{(1)}_{np} = 0$	$\psi^{(1)}_{np} = \mathrm{N}^{(1)}_n x$	Single antiparticle excitation state
$E^{(2)}_{np} = 0$	$\psi^{(2)}_{np} = \mathrm{N}^{(2)}_n y$	Single particle excitation state
$E^{(3)}_{np} = +w_n$	$\psi^{(3)}_{np} = \mathrm{N}^{(3)}_n\left(1 + \frac{m\ell^{-3}V}{\lambda_n \ell^{-2}V^{2/3} - w_n}xy\right)$	Pair excited state
where,	$w_n \equiv \left(\lambda_n^2\ell^{-4}V^{\frac{4}{3}} + m^2\ell^{-6}V^2\right)^{1/2}$	

- It follows the **inner product**, defined in the **holomorphic representation**:

$$\langle\psi_1, \psi_2\rangle = \int \overline{\psi_1}\,\psi_2\, e^{-x\bar{x}-y\bar{y}}\, dx\, d\bar{x}\, dy\, d\bar{y} \quad \Rightarrow \quad \hat{H}^{(\tau)}_{np} \text{ is } \textbf{self-adjoint} \text{ on } \mathscr{H}_{np}.$$

- **Discrete quantum geometry**: volume $v \sim a^3 \to \hat{V}$,
- Holonomy corrections replace classical $c \to \sin(\bar{\mu}c)$,
- Scalar field T as internal clock,
- **Quantum constraint** $\to$ Schrödinegr-like equation:

$$-i\hbar\partial_T \Psi_o(v) = \hbar\sqrt{\Theta}\Psi_o(v) =: \hat{H}_o\Psi_o(v) \qquad \text{(background evolution)}$$

where, $\Theta = -(2\ell^3/\hbar^2)\hat{H}^{(\tau)}_{\mathrm{grav}} \equiv \Theta_0 + \Theta_1$ defined by:

$$\Theta_0\Psi_o(v) = \frac{3\pi G}{4}\Big[(v+2)\sqrt{v(v+4)}\Psi_o(v+4) - 2v\Psi_o(v) + (v-2)\sqrt{v(v-4)}\Psi_o(v-4)\Big],$$

$$\Theta_1\Psi_o(v) = \frac{3\pi G}{2}\Big[\Big(\sin^2(\bar{\mu}\ell_o/2) - \bar{\mu}^2\ell_o^2/4\Big)v^2 - (\ell_o^2/9)\,(v/K)^{4/3}\Big]\Psi_o(v).$$

- Solution: $\Psi_o(v,T) = \sum_k c_k e_k(v) e^{i\omega_k T}$.
- **Physical Hilbert space:** $\Psi_o \in L^2(\bar{\mathbb{R}}, d\mu_{\mathrm{Bohr}}) \otimes L^2(\mathbb{R}, dT)$ in the $\ker(\hat{H}^{(\tau)}_{\mathrm{geo}})$, equipped with *inner product*:

$$\langle \Psi_o|\Psi'_o\rangle_\varepsilon = \sum_{v\in\mathscr{L}_\varepsilon} \overline{\Psi_o(v,T_0)}\,\Psi'_o(v,T_0).$$

$\mathscr{L}_\varepsilon = \mathscr{L}_{|\varepsilon|} \cup \mathscr{L}_{-|\varepsilon|}$, with $\mathscr{L}_{\pm|\varepsilon|}$ lattices of points $\{\pm|\varepsilon| + 4k;\ k \in \mathbb{Z}\}$ on the v-axis.

This provides a **well-defined quantum background** for perturbations.

- **Full quantum constraint:** Background + Fermion

$$\hat{H}_{\text{tot}}\Psi(T,v;x,y) = \left[\hat{H}_{\text{geo}}^{(\tau)} \otimes \mathbb{I} + \sum_{np} \hat{H}_{np}^{(\tau)}\right]\Psi(T,v;x,y) \approx 0,$$

where

$$\hat{H}_{\text{geo}}^{(\tau)} = -\frac{\hbar^2}{2\ell^3}\left(\partial_T^2 + \Theta\right),$$

$$\hat{H}_{np}^{(\tau)} = \left[\lambda_n \ell^{-2}\hat{V}^{2/3} \otimes (\widehat{x\bar{x}} + \widehat{y\bar{y}}) + m\ell^{-3}\hat{V} \otimes (\widehat{\bar{y}x} + \widehat{\bar{x}y})\right].$$

- **Challenges:**
  - Full system (geometry + fermions), $\Psi(T,v;x,y)$, is entangled.
  - Need approximation to extract physical predictions.

- **Strategy:** Separate heavy (geometry) and light (fermions) degrees of freedom:
  - **Test field approximation:** No backreaction; geometry state is the same:

$$\Psi_{np}(T,v;x,y) = \Psi_{np}^{o}(T,v) \otimes \psi_{np}(T,x,y).$$

  - **Born-Oppenheimer approximation:** With backreaction, geometry modified

$$\tilde{\Psi}_{np}(T,v;x,y) = \tilde{\Psi}_{np}^{o}(T,v) \otimes \psi_{np}(T,x,y).$$

This allows derivation of **effective dynamics** for fermion modes with/without backreaction.

Full constraint: Geometry + Fermion backreaction

$$\hat{H}_{\rm tot}\tilde{\Psi}^o_{np} = \left[-\frac{\hbar^2}{2\ell^3}(\partial_T^2 + \Theta) + \langle \hat{H}^{(T)}_{np}(v)\rangle_\psi\right]\tilde{\Psi}^o_{np} \approx 0, \qquad \Theta = -\frac{2\ell^3}{\hbar^2}\hat{H}^{(\tau)}_{\rm grav}(v)$$

Evolution equations:

- **Background:** Adding (operator-valued) fermion backreaction, $\hat{E}_n$, yields

$$-\partial_T^2\tilde{\Psi}^o_n(T,v) = \Big[\Theta + \Theta_n\Big]\tilde{\Psi}^o_n(T,v) =: \tilde{\Theta}_n\tilde{\Psi}^o_n(T,v).$$

Here, $\boxed{\Theta_n := -\frac{2\ell^3}{\hbar^2}\hat{E}_n(v)}$ where $\hat{E}_n(v) = \pm\sqrt{\frac{\lambda_n^2}{\ell^4}\hat{V}^{\frac{4}{3}} + \frac{m^2}{\ell^6}\hat{V}^2}$,

depending on whether the mode is in **vacuum** $(-)$ or **excited** $(+)$ state.

- **Fermion:** Tracing out the geometry DoF:

$$i\hbar\partial_T\psi_{np} = \Big[\lambda_n\ell^{-2}\langle\hat{H}_o^{-\frac{1}{2}}\hat{V}^{\frac{2}{3}}\hat{H}_o^{-\frac{1}{2}}\rangle(\widehat{x\bar{x}} + \widehat{y\bar{y}}) + m\ell^{-3}\langle\hat{H}_o^{-\frac{1}{2}}\hat{V}\hat{H}_o^{-\frac{1}{2}}\rangle(\widehat{\bar{y}x} + \widehat{\bar{x}\bar{y}})\Big]\psi_{np}.$$

$\langle\hat{O}\rangle$ is evaluated w.r.t. $\tilde{\Psi}^o_n = \Psi_o + \delta\Psi_n$: So,

$$\langle\hat{O}\rangle = \langle\hat{O}\rangle_o + \langle\hat{O}\rangle_n,$$

where, $\langle\hat{O}\rangle_n := \langle\Psi_o|\hat{O}|\delta\Psi_n\rangle + \langle\delta\Psi_n|\hat{O}|\Psi_o\rangle + \langle\delta\Psi_n|\hat{O}|\delta\Psi_n\rangle.$

Fermions modify the effective spacetime felt by other fields → **"dressed" metric**.

- **No backreaction.** The background geometry is *unperturbed*, Ψ_o; Result is a unique dressed metric for all modes:

$$d\tilde{s}_n^2 = -\bar{N}_T^2 dT^2 + \bar{a}^2 d\Omega_3^2,$$

$$\text{with} \quad \bar{N}_T = \ell^{-3} \langle \hat{H}_o^{-1/2} \hat{V} \hat{H}_o^{-1/2} \rangle_o \quad \text{and} \quad \bar{a} = \ell^{-1} \frac{\langle \hat{H}_o^{-1/2} \hat{V} \hat{H}_o^{-1/2} \rangle_o}{\langle \hat{H}_o^{-1/2} \hat{V}^{2/3} \hat{H}_o^{-1/2} \rangle_o}.$$

- **With backreaction.** Backreaction depends on mode n; Background is *perturbed*, $\Psi_o \to \tilde{\Psi}_n^o = \Psi_o + \delta\Psi_n$; Dressed metric is **mode-dependent**:

$$d\tilde{s}_n^2 = -\bar{N}_T^2 F_n^2 dT^2 + \bar{a}^2 G_n^2 d\Omega_3^2,$$

$$\text{with} \quad F_n = \left[\left(1 + \delta_n^{(1)}\right) \left(1 + \delta_n^{(2)}\right)^3 \right]^{\frac{1}{4}}, \qquad G_n = \left(\frac{1 + \delta_n^{(2)}}{1 + \delta_n^{(1)}} \right)^{\frac{1}{4}},$$

$$\text{and} \quad \delta_n^{(1)}(T) \equiv \frac{\langle \hat{H}_o^{-\frac{1}{2}} \hat{V}^{\frac{2}{3}} \hat{H}_o^{-\frac{1}{2}} \rangle_n}{\langle \hat{H}_o^{-\frac{1}{2}} \hat{V}^{\frac{2}{3}} \hat{H}_o^{-\frac{1}{2}} \rangle_o}, \qquad \delta_n^{(2)}(T) \equiv \frac{\langle \hat{H}_o^{-\frac{1}{2}} \hat{V} \hat{H}_o^{-\frac{1}{2}} \rangle_n}{\langle \hat{H}_o^{-\frac{1}{2}} \hat{V} \hat{H}_o^{-\frac{1}{2}} \rangle_o}.$$

Each fermion mode experiences a **distinct spacetime** → **"rainbow metric**."

Modified geometry evolution:

$$-\hbar^2\partial_T^2\tilde{\Psi}_n^o(T,v) = \left[\Theta + \Theta_n\right]\tilde{\Psi}_n^o(T,v), \quad \text{where} \quad \Theta_n := -\frac{2\ell^3}{\hbar^2}\hat{E}_n(v)$$

- General solution:

$$\tilde{\Psi}_n^o(T,v) = \sum_k c_k^{(n)} e_k^{(n)}(v) e^{i\omega_k T}. \tag{0.1}$$

- Gravitational sector is modified by introducing a mode-dependent potential, Θ_n, deforming the original (unperturbed) eigenfunctions as $e_k(v) \to e_k^{(n)}(v)$.

	Unperturbed background		**Backreacted background**
Eigenvalues & eigenfunctions	$\Theta e_k(v) = \omega_k^2\, e_k(v)$	$\to$	$\left[\Theta + \Theta_n\right] e_k^{(n)}(v) = (\omega_k^{(n)})^2\, e_k^{(n)}(v)$
Orthonormality of countable bases	$\langle e_k \vert e_{k'}\rangle = \delta_{k,k'}$	$\to$	$\langle e_k^{(n)} \vert e_{k'}^{(n)}\rangle = \delta_{k,k'}$
WdW regime	$\Theta \sim -\partial_v^2 + U_o$	$\to$	$\tilde{\Theta}_n \sim -\partial_v^2 + U_{\text{eff}}$
Effective potentials	$U_o(v)$	$\to$	$U_{\text{eff}}(v) = U_o(v) + \Theta_n(v)$

- The spectrum remains: Discrete, self-adjoint, and non-degenerate (under gap conditions), ensuring well-defined, singularity-free quantum evolution.

1. Shifted quantum bounce. The $\operatorname{sgn}(\Theta_n)$ determines the nature of the bounce:
   - **Vacuum state** ($\Theta_n > 0$): higher $U_{\text{eff}} \Rightarrow$ delayed, smoother bounce at higher density.
   - **Excited state** ($\Theta_n < 0$): lower barrier $\Rightarrow$ earlier, sharper bounce at lower density.
2. Dynamical control via particle creation:
   - Transition $\psi^{(0)}_{np} \to \psi^{(3)}_{np}$, driven by the time-dependent background (Parker-like particle production), injects energy $+2|E_n|$, causing negative jump $\Delta\Theta_n = -(4\ell^3/\hbar^2)|\hat{E}_n|$, that advances the bounce.
3. Rainbow of bounces. Since $E_n \sim \lambda_n/a \sim n$, then $|\Delta\Theta_n| \propto n$: Each mode n:
   - sees a distinct geometry, high-n modes: **stronger backreaction**;
   - has its own: V_n, T_n, transition sharpness $\Rightarrow$ **No universal quantum bounce!!**
4. Regime-dependent behavior:
   - **Deep Planckian regime** ($V \to V_{\text{b}}$): $\Theta_n \sim \pm\lambda_n \ell^{-2} \hat{V}^{2/3} \Rightarrow$ oscillatory near bounce, exponential decay otherwise.
   - **Large volumes**: mass dominates, $\Theta_n \sim \pm m \ell^{-3} \hat{V} \Rightarrow$ suppresses quantum effects, recovers semiclassical WKB behavior.

Fermionic backreaction—state- and mode-dependent—actively shapes quantum space-time beyond mean-field.

Fermions can influence the early universe in LQC in two main regimes:

- **Near the quantum bounce:** Massive fermions may be excited, producing pairs whose backreaction alters the energy density, breaks time symmetry, and induces asymmetric expansion with a preferred arrow of time.
- **Large volumes:** Massive fermions yield a nearly constant energy density acting as an effective cosmological constant:

$$\Lambda_{\text{eff}} \approx 8\pi G\rho_n = 8\pi G m\langle \hat{H}_o^{-1}\rangle_o.$$

  For a neutrino-mass scale, $m\sim 2\times 10^{-34} m_{\text{Pl}}$, we get $\Lambda_{\text{eff}} \sim 5\times 10^{-52}$. This is about 10^{70} times too large to explain dark energy (assuming $V_{\text{b}} \sim 10^{10} V_{\text{Pl}}$).
- **No standard particle** (including neutrinos) can reproduce the observed $\Lambda_{\text{obs}} \sim 5\times 10^{-122}$ under reasonable V_{b} assumptions.
- **Huge bounce volumes?** Achieving the observed Λ_{eff} would demand $V_{\text{b}} \sim 10^{52} V_{\text{Pl}}$—extreme fine-tuning far beyond expectations for a $k=+1$ FLRW universe ($V_{\text{b}} \lesssim 10^{20} V_{\text{Pl}}$). However, this issue can be reexamined in a **flat** universe.
- **Exotic fermions:** Matching observations requires an ultra-translight fermion with $m\sim 10^{-104} m_{\text{Pl}}$, absent from the Standard Model or current dark matter candidates.

Key Findings:

- Complete analysis of **fermions in closed LQC**;
- Derived dressed metric for the fermion modes;
- Derived **mode-dependent backreaction**;
- Found **rainbow metric**: each fermion mode sees its own geometry;
- Bounce properties depend on mode frequency ⇒ "rainbow of bounces".
- Evolution remains **unitary and singularity-free**.

Fermions are not just spectators—they **reshape quantum spacetime**.

Open Directions:

- Extend to **massive fermions** beyond adiabatic regime
- Include **self-interactions** or gauge fields
- Study **particle creation** and reheating
- Compare with **Hartle-Hawking** or **D'Eath-Halliwell** proposals
- Explore **observational consequences** in early universe.

Thank you!