Cauchy–Riemann geometry in general relativity: a brief survey

In memory of Jerzy Lewandowski (1959-1924)

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Overview

- Cauchy-Riemann (CR) geometry: Study of real hypersurfaces in $\mathbf{C}^n$
- CR geometry and Lorentzian (conformal) geometry (General relativity) both arose at the beginning of the twentieth century with the work of Poincaré, Cartan...
- However, their interaction only slowly understood from the 1960ies to the mid-1970ies.
- Hinted in the works by Robinson, Trautman, Kerr, Newman, Penrose... in their investigation of the geometry of light rays
- Two schools:
  - Oxford school: Penrose, Sparling, LeBrun, Mason... (twistorial ideas)
  - Warsaw school: Trautman, Tafel, Lewandowski, Nurowski... (CR realisability)

Jurek's contributions

- Thesis: Zastosowanie geometrii Cauchy–Riemanna do badania pola grawitacyjnego
- On the Fefferman class of metrics associated with a three-dimensional CR space. *Lett. Math. Phys.* **15** (1988), no. 2, 129-135.
- (with Paweł Nurowski) Algebraically special twisting gravitational fields and CR structures. *Classical Quantum Gravity* **7** (1990), no. 3, 309-328.
- (with Paweł Nurowski, Jacek Tafel) Einstein's equations and realizability of CR manifolds. *Classical Quantum Gravity* **7** (1990), no. 11, L241-L246.
- (with Paweł Nurowski) Cartan's chains and Lorentz geometry. *J. Geom. Phys.* **7** (1990), no. 1, 63-80.
- Twistor equation in a curved spacetime. *Classical Quantum Gravity* **8** (1991), no. 1, L11-L17.
- (with Paweł Nurowski, Jacek Tafel) Algebraically special solutions of the Einstein equations with pure radiation fields. *Classical Quantum Gravity* **8** (1991), no. 3, 493-501.
- (with Jacek Tafel, Paweł Nurowski) Pure radiation field solutions of the Einstein equations. *Classical Quantum Gravity* **8** (1991), no. 4, L83-L88.
- (with C. Denson Hill, Paweł Nurowski) Einstein's equations and the embedding of 3-dimensional CR manifolds. *Indiana Univ. Math. J.* **57** (2008), no. 7, 3131-3176.

Robinson congruence

- Minkowski space $\mathbb{M} = \{u, z, \bar{z}, r\}$ with null v.f. $\widetilde{k} = \frac{\partial}{\partial r}$:

$$\widetilde{g} = 2\kappa \mathrm{d}r + 2(r^2 + 1)\theta^1 \overline{\theta}^{\bar{1}}$$

$$\kappa = \widetilde{g}(\widetilde{k}, \cdot) = \mathrm{d}u - \mathrm{i}\bar{z}\mathrm{d}z + \mathrm{i}z\mathrm{d}\bar{z}\,, \qquad \theta^1 = \mathrm{d}z\,.$$

- Twisting shearfree congruence of null geodesics $\widetilde{\mathcal{K}}$ generated by $\widetilde{k}$:

$$\mathcal{L}_{\widetilde{k}}\widetilde{g}|_{\langle\widetilde{k}\rangle^\perp} \propto \widetilde{g}|_{\langle\widetilde{k}\rangle^\perp}\,, \qquad \kappa \wedge \mathrm{d}\kappa \neq 0\,.$$

- Robinson structure $(\widetilde{N}, \widetilde{K})$: involutive totally null complex 2-plane (α-plane) distribution

$$\widetilde{N} = \mathrm{Ann}(\kappa, \theta)\,, \qquad \widetilde{N} \cap \overline{\widetilde{N}} = \mathbf{C} \otimes \langle\widetilde{k}\rangle\,, \qquad [\widetilde{N}, \widetilde{N}] \subset \widetilde{N}\,.$$

- Spinor field $\widetilde{\xi}^{A'}$: $\widetilde{N} = \langle \widetilde{o}^A \widetilde{\xi}^{A'}, \widetilde{\iota}^A \widetilde{\xi}^{A'} \rangle$ for any spinor frame $\widetilde{o}^A, \widetilde{\iota}^A$
  $\widetilde{\xi}^{A'}\widetilde{\xi}^{B'}\nabla_{AA'}\widetilde{\xi}_{B'} = 0 \iff \widetilde{N}$ involutive
- Contact Cauchy–Riemann (CR) structure $H^{(1,0)}$ on the leaf space $\mathcal{M} = \{u, z, \bar{z}\}$ of $\mathcal{K}$:

$$\overline{H^{(1,0)}} = H^{(0,1)} := \mathrm{Ann}(\kappa, \theta)\,, \qquad H = \Re(H^{(1,0)}) := \mathrm{Ann}(\kappa)\,,$$

  Hyperquadric $\mathcal{M} = \{(z, w) \in \mathbf{C}^2 : \Im(w) = |z|^2\}$
- $F = \kappa \wedge \theta$ satisfies vacuum Maxwell equations: $\mathrm{d}F = \mathrm{d} \star F = 0$

The Kerr congruence

- Kerr metric (1963): Petrov type D vacuum spacetime $\widetilde{\mathcal{M}} = \{u, \vartheta, \phi, r\}$ with parameters a and m:
  $$\widetilde{g} = 2\kappa \left(\mathrm{d}r + a\sin^2\vartheta\mathrm{d}\phi + \left(\frac{mr}{r^2+a^2\cos^2\vartheta} - \frac{1}{2}\right)\kappa\right) + 2(r^2 + a^2\cos^2\phi)\theta^1\overline{\theta}^{\bar{1}} \,,$$
  $$\kappa = \mathrm{d}t + a\sin^2\vartheta\mathrm{d}\phi \,, \qquad \theta^1 = \mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\mathrm{d}\phi \,.$$
- Twisting shearfree congruence generated by $\widetilde{k} = \frac{\partial}{\partial r}$
- Robinson structure: $\widetilde{N} = \mathrm{Ann}(\kappa, \theta)$
- Contact CR structure $H^{(1,0)}$ on the leaf space $\mathcal{M} = \{u, \vartheta, \phi\}$ of $\mathcal{K}$:
  $$H := \mathrm{Ann}(\kappa) \,, \qquad H^{(0,1)} := \mathrm{Ann}(\kappa, \theta^1) \,.$$
- Note $\theta^1 = \mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\mathrm{d}\phi$ satisfies $\theta^1 \wedge \mathrm{d}\theta^1 = 0$, i.e.
  $$\theta^1 \wedge \mathrm{d}z = 0$$
  for some smooth $z : \mathcal{M} \to \mathbf{C}$ s.t. $X(z) = 0$ for any $X \in \Gamma H^{(0,1)}$
- z known as Kerr coordinate (or CR function)
- Two CR functions $\Rightarrow$ $(\mathcal{M}, H^{(1,0)})$ realisable as a real hypersurface in $\mathbf{C}^2$.

Classical results of the Golden Age

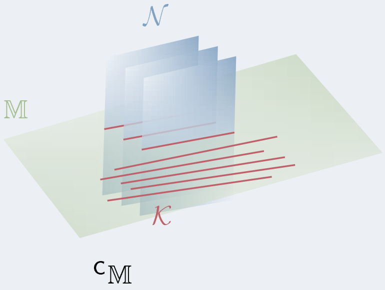
- Conformal Lorentzian 4-manifold $\overset{\text{shearfree cong.}}{\longleftrightarrow}$ CR three-manifold
- Mariot (1954), Robinson (1961) Theorem: Solutions to the vacuum Maxwell equations ($\mathrm{d}F = \mathrm{d} \star F = 0$) give rise to shearfree congruences of null geodesics. 'Conversely', any analytic shearfree congruence gives rise to a solution to the vacuum Maxwell equations. (See also Tafel (1985), Holland-Sparling (2013))
- Goldberg-Sachs (1962) Theorem: For a spacetime metric that is a solution to the vacuum EFE, the Weyl tensor is algebraically special if and only if there exists a shearfree congruence of null geodesics.
- Kerr Theorem (Penrose (1967)): Any analytic shearfree congruence of null geodesics determines and is determined a single holomorphic function of three complex variables.

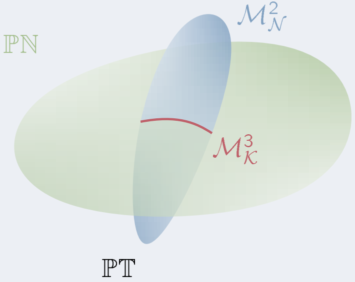
Kerr surfaces in Twistor space

Kerr theorem (Penrose (1967))

Any analytic shearfree null geodesic congruence in Minkowski space $\mathbb{M}$ locally gives rise to a complex (Kerr) surface in TC (2023) space $\mathbb{PT}$. Conversely, any such congruence arises in this way.

- Twistor space $\mathbb{PT} \cong \mathbf{C}\mathbb{P}^3$: space of α-planes in ${}^{\mathbf{C}}\mathbb{M}$
- Flat CR manifold $\mathbb{PN} = S^2 \times S^3/\mathbf{Z}_2 \subset \mathbb{PT}$: space of null geodesics
- Shearfree null geodesic congruence $\mathcal{K} = \mathbb{M} \cap \mathcal{N}$ where $\mathcal{N}$ is a foliation by α-planes





Lift of CR structures

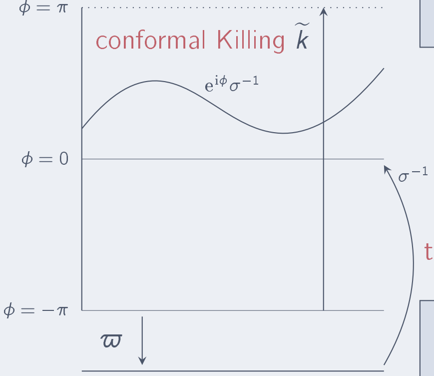
- Lewandowski-Nurowski (1990):
  CR manifold $(\mathcal{M}, H^{(1,0)})$, adapted coframe $(\theta, \theta^1, \overline{\theta}^{\bar{1}})$: $\mathrm{d}\theta = \mathrm{i}\theta^1 \wedge \overline{\theta}^{\bar{1}}$
  Trivial bundle $\widetilde{\mathcal{M}} = \mathcal{M} \times \mathbf{R}$ with fibre coordinate ϕ
- Choose any nowhere-vanishing function $\widetilde{\Omega}$ and semi-basic one-form $\widetilde{\lambda}$ on $\widetilde{\mathcal{M}}$, i.e.
  $$\widetilde{\lambda} = \mathrm{d}\phi + \widetilde{\lambda}_1\theta^1 + \widetilde{\lambda}_{\bar{1}}\overline{\theta}^{\bar{1}} + \widetilde{\lambda}_0\theta\,, \quad \text{for some functions } \widetilde{\lambda}_1, \widetilde{\lambda}_0 \text{ on } \widetilde{\mathcal{M}}.$$
  Then
  $$\widetilde{g} = \widetilde{\Omega}^2 \left(4\theta\widetilde{\lambda} + 2\theta^1\overline{\theta}^{\bar{1}}\right),$$
  is a metric on $\widetilde{\mathcal{M}}$ and $\widetilde{k} = \frac{\partial}{\partial\phi}$ generates a twisting shearfree congruence of null geodesics.
- Impose (subsystem) of vacuum EFE: ϕ-dependence integrated out
  $$\widetilde{\Omega}^2 = \mathrm{e}^{\varphi}\sec^2(\phi + \psi)\,, \qquad \widetilde{\lambda}_1 = \lambda_1^{(0)} + \lambda_1^{(-2)}\mathrm{e}^{-2\mathrm{i}\phi}\,,$$
  $$\widetilde{\lambda}_0 = \lambda_0^{(4)}\mathrm{e}^{4\mathrm{i}\phi} + \lambda_0^{(2)}\mathrm{e}^{2\mathrm{i}\phi} + \lambda_0^{(0)} + \lambda_0^{(-2)}\mathrm{e}^{-2\mathrm{i}\phi} + \lambda_0^{(-4)}\mathrm{e}^{-4\mathrm{i}\phi}$$
  where φ, ψ, $\lambda_1^{(i)}$, $\lambda_0^{(i)}$ are functions on $\mathcal{M}$ satisfying PDEs.
- Hill-Lewandowski-Nurowski (2008): smooth metric $\widetilde{\Omega}^{-2}\widetilde{g}$ on S^1-bundle. Any relation to Fefferman's construction?

Fefferman space Fefferman (1976), Lee (1986), Čap-Gover (2008)

$(\mathcal{M}, H^{(1,0)})$ CR 3-manifold with canonical bundle $\mathcal{C} := \Lambda^2 \mathrm{Ann}(H^{(0,1)})$

Density bundle $\mathcal{E}(1,0) := \mathcal{C}^{-\frac{1}{3}}$, $\mathcal{E}(w,w') := \mathcal{E}(1,0)^w \otimes \overline{\mathcal{E}(1,0)^{w'}}$

$\widetilde{\mathcal{M}} := \mathcal{E}^*(-1,0)/\mathbf{R}_{>0}$



$(\mathcal{M}, H^{(1,0)})$, $H = \Re(H^{(1,0)})$

trivialisation σ

Fefferman metric

$$\widetilde{g}_\theta := 4\varpi^*\theta \odot \left(\widetilde{\omega}^\theta - \tfrac{1}{3}\mathrm{P}^\theta\theta\right) + \varpi^* h^\theta$$

$$\theta \mapsto \widehat{\theta} = e^{\varphi}\theta \quad \rightsquigarrow \quad \widetilde{g}_{\widehat{\theta}} = e^{\varphi}\widetilde{g}_\theta$$

$\rightsquigarrow$ conformal structure $\widetilde{\mathbf{c}}$

Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k})$

$\theta \in \Gamma(\mathrm{Ann}(H))$, Levi form $h^\theta := d\theta|_H$

Webster connection ∇^θ on $T\mathcal{M}$

Induced connection $\widetilde{\omega}^\theta$ on $\widetilde{\mathcal{M}}$

Webster–Schouten scalar P^θ

Algebraically special spacetimes and Fefferman's conformal structure

- Can we find conformally invariant conditions such that a spacetime with a shearfree congruence is locally conformally to a conformal structure on Fefferman's circle bundle?

Theorem (TC (2023))

Let $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}})$ be an algebraically special (conformal) spacetime with repeated gravitational principal null direction $\langle \widetilde{k} \rangle$, i.e. $\widetilde{\mathsf{W}}(\widetilde{k}, \widetilde{v}, \widetilde{k}, \cdot) = 0$ for any $\widetilde{v} \in \Gamma(\langle \widetilde{k} \rangle^{\perp})$. Suppose that the Bach tensor satisfies $\widetilde{\mathsf{B}}(\widetilde{k}, \widetilde{k}) = 0$. Then $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}})$ is locally conformally isometric to a perturbed Fefferman conformal space, i.e. any metric in $\mathbf{c}$ is conformally related to

$$\widetilde{g}_{\theta, \widetilde{\xi}} = \widetilde{g}_{\theta} + 4\theta \odot \widetilde{\xi}$$

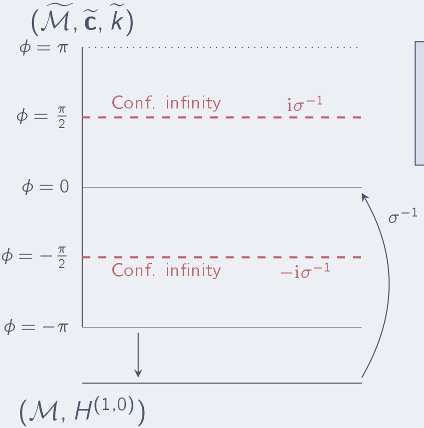
for some Fefferman metric $\widetilde{g}_{\theta}$ and semi-basic one-form $\widetilde{\xi}$. In addition, the components of $\widetilde{\xi}$ have Fourier expansions

$$\xi_1 = \xi_1^{(0)} + \xi_1^{(-2)} \mathrm{e}^{-2\mathrm{i}\phi},$$

$$\xi_0 = \xi_0^{(4)} \mathrm{e}^{4\mathrm{i}\phi} + \xi_0^{(2)} \mathrm{e}^{2\mathrm{i}\phi} + \xi_0^{(0)} + \xi_0^{(-2)} \mathrm{e}^{-2\mathrm{i}\phi} + \xi_0^{(-4)} \mathrm{e}^{-4\mathrm{i}\phi}.$$

Pure radiation metrics

- Lewandowski (1988): Any Fefferman conformal structure that contains a (local) Einstein metric must be conformally flat.
- But how about perturbed Fefferman spaces?
- TC (2023): Pure radiation metric as perturbed Fefferman metric:



pure radiation metric

$$\widetilde{g} = \sec^2\phi \cdot \widetilde{g}_{\theta,\widetilde{\xi}}, \qquad \phi \neq \pm\tfrac{\pi}{2}$$

$$\widetilde{\mathrm{Ric}}^{\widetilde{g}} = \Lambda\widetilde{g} + \Phi\theta^2$$

↑ CR PDEs

perturbed Fefferman metric $\widetilde{g}_{\theta,\widetilde{\xi}}$

↑

density σ
+ CR data $\boldsymbol{\xi}_0^{(i)}$, $\boldsymbol{\xi}_1^{(i)}$

Some CR analysis

- Subsystems of vacuum EFE reduces to CR PDEs that have clear interpretations in terms of the analytic properties of CR geometry
- Kerr (1963), Lewandowski-Nurowski-Tafel (1990), Mason (1998): $(\mathcal{M}, H^{(1,0)})$ must admit a complex-valued 1-form $\omega \in \Gamma(\mathrm{Ann}(H^{(0,1)}))$ such $\omega \wedge \mathrm{d}\omega = 0$.

Hill–Lewandowski–Nurowski's Lemma (2008)

$(\mathcal{M}, H^{(1,0)})$ admits a (local) CR function f, i.e. $X(f) = 0$ for all $X \in \Gamma(H^{(1,0)})$ if and only if there exists a complex-valued 1-form $\omega \in \Gamma(\mathrm{Ann}(H^{(0,1)}))$ such $\omega \wedge \mathrm{d}\omega = 0$.

Theorem (TC (2023))

$(\mathcal{M}, H^{(1,0)})$ locally admits a CR function if and only if there exists a solution λ_α to the Webster–Weyl equation:

$$\nabla_\alpha \lambda_\beta - \mathrm{i}\lambda_\alpha \lambda_\beta - A_{\alpha\beta} = 0 \,. \tag{1}$$

Here, ∇ is the Webster connection with Webster torsion $A_{\alpha\beta}$.

- Think of (1) as a CR analogue of Einstein–Weyl equation (or quasi-Einstein equation - another of Jurek's contributions!)

Realisable CR structures and Einstein metrics

Theorem (Lewandowski-Nurowski-Tafel (1990), Hill–Lewandowski–Nurowski (2008))

If a CR 3-manifold $(\mathcal{M}, H^{(1,0)})$ lifts to an Einstein metric on $\mathcal{M} \times \mathbf{R}$ then it is realisable as a real hypersurface in $\mathbf{C}^2$.

- Lewy (1957), Niremberg (1974), Jacobowitz-Trèves (1982)...: Not every CR three-manifold is realisable!
- Jacobowitz (1987): $(\mathcal{M}, H^{(1,0)})$ is locally embeddable if and only if it admits a transverse complex-valued vector field ℓ preserving the CR structure, i.e. $\mathcal{L}_\ell H^{(1,0)} \subset H^{(1,0)}$.
- Curry-Ebenfelt (2019), TC (2023): Equivalently, there exists a density $\sigma \in \Gamma(\mathcal{E}(1,1))$ that satisfies

$$\nabla_\alpha \nabla_\beta \sigma + \mathrm{i} A_{\alpha\beta}\sigma = 0\,. \tag{2}$$

  (1) is the non-linear analogue of (2)
- See also: Tafel (1985), Jacobowitz (1987,2020), Holland-Sparling (2013)... for further results regarding CR embeddability.

Story in higher dimensions

- Hughston-Mason (1988): $2m$-dimensional generalisation of the Kerr and Robinson theorems in the language of pure spinors:
$$\nabla_X \xi \propto \xi \,, \qquad \text{for all v.f. } X \text{ s.t. } X \cdot \xi = 0,$$
i.e. $N_\xi := \{X \in \Gamma({}^{\mathbb{C}}T\mathcal{M}) : X \cdot \xi = 0\}$ involutive maximally totally null
- Mason-TC (2010): Conformal Killing–Yano two-forms and applications to Kerr–NUT–Ad(S) metrics
- Nurowski-Trautman (2005), Fino-Leistner-TC (2023): Nearly Robinson manifold: conformal Lorentzian manifold admitting a null geodesic congruences (not necessarily shearfree) whose leaf space is an almost CR manifold
- TC (2019): Obtained all solutions to vacuum EFE for Lorentzian manifolds of dim $2m > 4$ admitting a twisting shearfree congruence of null geodesics with a weak algebraically degenerate Weyl tensor. Includes Taub–NUT–(A)dS metrics
- TC (2025): Formulation in terms of perturbed Fefferman spaces

Hypersurface twistors Penrose, Sparling, LeBrun, Mason...

- Another aspect of the interaction between CR geometry and general relativity: The space of light rays of a conformal spacetime $(\widetilde{M}, \widetilde{\mathbf{c}})$. Following Mason (1985):
- Projective spinor bundle $\mathbb{P}\widetilde{\mathcal{S}}^+ \to \widetilde{\mathcal{M}}$ with S^2-fibers: at a point $x \in \widetilde{\mathcal{M}}$, a point $z \in \mathbb{P}\widetilde{\mathcal{S}}^+_x$ corresponds to an α-plane $\widetilde{N}_x(z)$ in ${}^{\mathbf{C}}T_x\widetilde{\mathcal{M}}$
- Twistor distribution D is the (conformally invariant) tautological complex rank-three distribution on the total space of $\mathbb{P}\mathcal{S}^+$: Any α-plane $\widetilde{N}_x(z)$ lifts to a horizontal two-plane $\mathrm{N}_{(x,z)}$ in $\mathbb{P}\mathcal{S}^+_{(x,z)}$ independently of the choice of the Levi-Civita connection. Then $\mathrm{D}_{(x,z)} = \mathrm{N}_{(x,z)} \oplus \langle \frac{\partial}{\partial \bar{z}} \rangle$.
- $\mathrm{D} \cap \overline{\mathrm{D}} = \mathbf{C} \otimes \mathrm{K}$ where K is tangent to the null geodesic spray $\mathcal{K}$.
- Quotient $\mathbb{P}\mathcal{N} = \mathbb{P}\widetilde{\mathcal{S}}^+/\mathcal{K}$ is the real five-dimensional space of null geodesics
- D is involutive if and only if $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}})$ is conformally flat!
- But can be fixed...

- Choose a hypersurface $\widetilde{\mathcal{H}}$ in $\widetilde{\mathcal{M}}$
- View the restriction $\mathbb{P}\widetilde{\mathcal{S}}^{+}|_{\widetilde{\mathcal{H}}}$ as a cross-section of $\mathbb{P}\widetilde{\mathcal{S}}^{+} \to \mathbb{P}\mathcal{N}$
- Now, $\mathrm{D}|_{\widetilde{\mathcal{M}}}$ is an involutive complex rank-two distribution, so defines a CR structure on $\mathbb{P}\widetilde{\mathcal{S}}^{+}|_{\widetilde{\mathcal{H}}}$
- Its pull-back to $\mathbb{P}\mathcal{N}$ defines a CR structure of Levi signature $(1,1)$
- (Small) drawback: CR structure depends on the choice of hypersurface...
- However, for algebraically special spacetimes satisfying (subsystems of) the vacuum EFE with twisting shearfree congruences, can use the perturbed Fefferman approach:
  - two distinguished hypersurfaces available (conformal infinities);
  - a distinguished section of $\mathbb{P}\mathcal{S}^{+}$ (determined by the congruence) that intersect $\mathbb{P}\mathcal{S}^{+}|_{\widetilde{\mathcal{H}}}$ in a CR 3-manifold.

Thank you for your attention!