Construction of hyperboloidal initial data without logarithmic singularities

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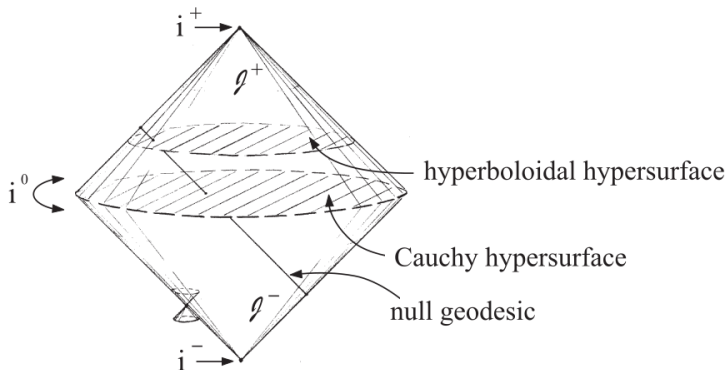**Jerzy Lewandowski Memorial Conference**
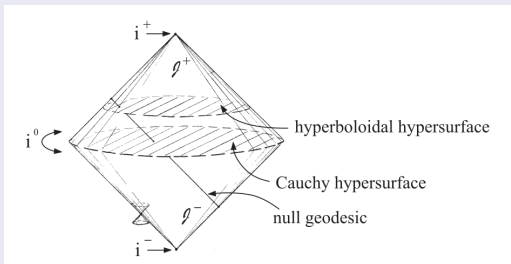Banach Center, Warszawa, Poland
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the asymptotic behavior of the gravitational field of an isolated system resembles that of Minkowski space

- There exists a smooth manifold $\widetilde{M} = M \cup \partial M$ with boundary,
- and a smooth Lorentz metric $\widetilde{g} = \Omega^2 g$ on $\widetilde{M}$
- such that $\Omega = 0$, $d\Omega \neq 0$ on ∂M
  - The **degree of smoothness** of the involved structures **is critical**.

Asymptotically simple spacetimes & asymptotically hyperboloidal data

- **Friedrich:** Solutions to the **hyperboloidal initial value formulation of his conformal field equations** are asymptotically simple in the future.
    - Hyperboloidal initial data – sufficiently and close to Minkowskian hyperboloidal data – develop into solutions which admit smooth conformal extensions containing a regular point i^+ that represents future time-like infinity.

the asymptotic behavior of the gravitational field of an isolated system resembles that of Minkowski space [Credit: H.Friedrich]

- An **asymptotically hyperboloidal vacuum** data set (Σ, h_{ij}, K_{ij}) is such that
    - Σ is the interior of a **compact manifold with boundary** $\Sigma \cup \partial\Sigma$
    - the **trace** $K = K_{ij}h^{ij}$ of K_{ij} is **bounded away from zero** near $\partial\Sigma$
    - if ω is a defining function for $\partial\Sigma$ then $\omega^2 h_{ij}$ and $\omega \overset{\circ}{K}_{ij}$ **extend regularly** to $\partial\Sigma$

Constraint equations

Einstein equations: $E_{ab} = G_{ab} - \mathscr{G}_{ab} = 0$ $n^a n^b E_{ab} = 0$ & $\pi_i{}^a n^b E_{ab} = 0$

- in the vacuum case ($\mathscr{G}_{ab} = 0$) for (h_{ij}, K_{ij}) on Σ

$$\boxed{{}^{(3)}R + \left(K_{ij}h^{ij}\right)^2 - K_{ij}K^{ij} = 0 \quad \& \quad D^i\left[K_{ij} - h_{ij}\left(K_{ef}h^{ef}\right)\right] = 0} \quad (D_k h_{ij} = 0)$$

- It is an underdetermined system, 4 equations for the 12 variables: $\boxed{(h_{ij}, K_{ij})}$

Conformal method (Lichnerowicz 1944, York 1972)

$$h_{ij} = \phi^4\, \widetilde{h}_{ij}, \qquad \overset{\circ}{K}_{ij} = K_{ij} - \tfrac{1}{3}\, h_{ij}K = \phi^{-2}\, \widetilde{K}_{ij}, \qquad \widetilde{K}_{ij} = CKO[X]_{ij} + \widetilde{K}_{ij}^{[TT]}$$

$$(h_{ij}, K_{ij}) \quad \longleftrightarrow \quad (\phi, \widetilde{h}_{ij}; K, X_i, \widetilde{K}_{ij}^{[TT]})$$

the constraints form a quasilinear elliptic system for (ϕ, X_i)

Evolutionary method (I.R. 2015)

Σ is foliated by topological two-spheres & applying an ADM type decomposition

$$h_{ij} \longleftrightarrow \widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij} \quad \& \quad K_{ij} \longleftrightarrow \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}_{ij}\, [= \overset{\circ}{\mathbf{K}}_{ij} + \tfrac{1}{2}\, \mathbf{K}\, \widehat{\gamma}_{ij}]$$

using a complex null dyad q^i: $q_{ij} = q_{(i}\bar{q}_{i)}$ $(h_{ij}; K_{ij}) \quad \longleftrightarrow \quad (\widehat{\mathbf{N}}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \overset{\circ}{\mathbf{K}}_{qq})$

the constraints form a parabolic-hyperbolic system for $(\widehat{\mathbf{N}}, \mathbf{k}, \mathbf{K})$

Andersson & Chruściel '93, '94, '96: log-terms become part of the discussions

- They proved that **even if the free data is smooth** on $\widetilde{\Sigma} = \Sigma \cup \partial\Sigma$,

  (1) **in general**, the constrained fields have poly-logarithmic expansions in $\omega \sim \rho^{-1}$, where ρ denotes the "distance" from the isolated system

  $$\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1\omega + \mathcal{C}_2\omega^2 + \dots \qquad \mathcal{C} = \mathcal{C}_0 + \sum_{i=1}^{\infty} \omega^i \left[\mathcal{C}_i + \sum_{j=1}^{\mathcal{N}_j} \mathcal{C}_{i,j}^{[log]} \log^j \omega \right]$$

  $(n \geq 1)$ $\boxed{\omega \log^n \omega \to 0 \quad \textbf{but} \quad \partial_\omega(\omega \log^n \omega)}$ blows up

  (2) **non-generic cases:** the initial data can be smooth (free of log-terms) on $\widetilde{\Sigma}$ if the relations

  $$\widetilde{K}_{ra}^{[log]}\big|_{\partial\Sigma} = 0 \text{ in } \widetilde{K}_{ab} = \widetilde{K}_{ab}^{[C^\infty]} + \Omega^2 \log\Omega \cdot \widetilde{K}_{ab}^{[log]} \quad \& \quad \boxed{(\widetilde{\mathbf{K}}_{ab}^{\circ} - \widetilde{\widetilde{K}}_{ab}^{\circ})\big|_{\partial\Sigma} = 0}$$

  (derived from the free data) **hold on** $\partial\Sigma$.

Facing the problems:

- These conclusions were disappointing because they implied that the initial data constructed by the conformal method is **generally not regular enough** for use in Friedrich's existence theorems. $[\widetilde{\nabla}_e d_{abc}{}^e = 0 \quad \& \quad d_{abc}{}^e = \Omega^{-1} C_{abc}{}^e]$

- **If the initial data involves log-terms**, then **the development will also contain**

- How can then the metric be decomposed into a sufficiently smooth non-physical metric and a conformal factor? the concept of asymptotically simple spacetimes?

Important new results by Beyer and Ritchie [CQG,**39**,145012 (2022)]

- They assumed that there exist smooth global solutions to the **parabolic-hyperbolic form of the constraints** on a "hyperboloidal initial data surface" Σ.

- They also assumed that these solutions **extend regularly up to order** 4 **and** 3, respectively, to $\partial\Sigma$
  - $\widehat{\mathbf{N}}, \mathbf{K} \in C^4\big([0,\omega_0), C^\infty(\mathscr{S}^2)\big)$
  - $\mathbf{k} \in C^3\big([0,\omega_0), C^\infty(\mathscr{S}^2)\big)$

- Then, using an **impressive Fuchsian-equation based argument** they showed that **the constrained variables extend smoothly** to $\partial\Sigma$, whence such solutions are free of all log-terms.

- **parabolic-hyperbolic form of the constraints** $\checkmark$
  - BUT **even stronger** assumptions than used by Andersson and Chruściel.
  - They restricted not only the free data (that went largely uncommented)
  - **but** also **the constrained fields** $\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k}$ **!!!**

The strategy we used:

Asymptotically hyperboloidal data: $\qquad (h_{ij}; K_{ij}) \longleftrightarrow (\widehat{\mathbf{N}}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \overset{\circ}{\mathbf{K}}_{qq})$

- Recall first that a data set $(\widehat{\mathbf{N}}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \overset{\circ}{\mathbf{K}}_{qq})$ can only be **asymptotically hyperboloidal** if the following falloff conditions hold: $\omega \sim \rho^{-1}$

$$\widehat{\mathbf{N}} = \widehat{\mathbf{N}}_1 \omega + \mathcal{O}(\omega^2) \qquad \mathbf{K} - 2\boldsymbol{\kappa} = \mathcal{O}(\omega) \qquad \mathbf{k} = \mathcal{O}(1)$$
$$\mathbf{a} = \omega^{-2} + \mathcal{O}(\omega^{-1}) \qquad \mathbf{b} = \mathcal{O}(\omega^{-1}) \qquad \mathbf{N} = \mathcal{O}(\omega) \qquad \overset{\circ}{\mathbf{K}}_{qq} = \mathcal{O}(\omega^{-1})$$

The free data is assumed to be smooth on $\Sigma \cup \partial\Sigma$: $\qquad C^\infty\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$

$$\mathbf{N} = \mathbf{N}_1\,\omega + \mathbf{N}_2\,\omega^2 + \mathcal{O}(\omega^3)$$
$$\mathbf{a} = \omega^{-2} + \mathbf{a}_{(-1)}\,\omega^{-1} + \mathbf{a}_0 + \mathbf{a}_1\,\omega + \mathbf{a}_2\,\omega^2 + \mathcal{O}(\omega^3)$$
$$\mathbf{b} = \mathbf{b}_{(-1)}\,\omega^{-1} + \mathbf{b}_0 + \mathbf{b}_1\,\omega + \mathbf{b}_2\,\omega^2 + \mathcal{O}(\omega^3)$$
$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1\,\omega + \boldsymbol{\kappa}_2\,\omega^2 + \mathcal{O}(\omega^3)$$
$$\overset{\circ}{\mathbf{K}}_{qq} = \overset{\circ}{\mathbf{K}}_{qq(-1)}\,\omega^{-1} + \overset{\circ}{\mathbf{K}}_{qq0} + \overset{\circ}{\mathbf{K}}_{qq1}\,\omega + \overset{\circ}{\mathbf{K}}_{qq2}\,\omega^2 + \mathcal{O}(\omega^3)$$

Use the most generic poly-logarithmic form of the constrained fields $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$:

$$\widehat{\mathbf{N}} = \sum_{i=1}^\infty \omega^i \Big[\,\widehat{\mathbf{N}}_i + \sum_{j=1}^{\mathcal{N}_j} \widehat{\mathbf{N}}_{i,j}^{[log]} \log^j \omega\,\Big], \qquad \mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^\infty \omega^i \Big[\,\mathbf{K}_i + \sum_{j=1}^{\mathcal{N}_j} \mathbf{K}_{i,j}^{[log]} \log^j \omega\,\Big]$$
$$\mathbf{k} = \mathbf{k}_0 + \sum_{i=1}^\infty \omega^i \Big[\,\mathbf{k}_i + \sum_{j=1}^{\mathcal{N}_j} \mathbf{k}_{i,j}^{[log]} \log^j \omega\,\Big], \text{ where } \widehat{\mathbf{N}}_1 = \boldsymbol{\kappa}_0^{-1}, \mathbf{K}_0 = 2\boldsymbol{\kappa}_0, \mathbf{k}_0 = \boldsymbol{\kappa}_0^{-1}\eth\boldsymbol{\kappa}_0$$

Our first main result

Theorem I:

- Choose **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$ on Σ such that it **satisfies the asymptotically hyperboloidal falloff conditions** and **it is smooth on** $\Sigma \cup \partial\Sigma$.

- Suppose that $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are **smooth solutions** of the parabolic-hyperbolic form of the constraints on Σ.

- $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are also assumed to possesses the most generic poly-logarithmic expansion near $\partial\Sigma$ as indicated above.

- Then the corresponding asymptotically hyperboloidal initial data set **admits well-defined Bondi energy and angular momentum if and only if** all coefficients of **the logarithmic terms vanish up to order four and three** for $\widehat{\mathbf{N}}, \mathbf{K}$ and $\mathbf{k}$, respectively, and, **in addition,**

$$\overset{\circ}{\mathbf{K}}_{qq(-1)} = 0, \quad \mathbf{b}_{(-1)} = 0, \quad \boldsymbol{\kappa}_1 = 0.$$

The finiteness of the Bondi energy:

The Bondi energy can be given as the $\rho \to \infty$ limit of the Hawking (mass) energy

$$m_H = \sqrt{\frac{\mathcal{A}}{16\pi}} \left(1 + \frac{1}{16\pi} \int_{\mathscr{S}_\rho} \underbrace{(\mathbf{K}^2 - \overset{\star}{\mathbf{K}}{}^2 \widehat{\mathbf{N}}^{-2})}_{= \, \Theta^{(+)} \Theta^{(-)}} \sqrt{\mathbf{d}}\, \overset{\circ}{\boldsymbol{\epsilon}} \right)$$

$\mathcal{A}, \sqrt{\mathbf{d}} \sim \rho^2$, $\overset{\star}{\mathbf{K}} = \partial_\rho \log[\sqrt{\mathbf{d}}] \sim \rho^{-1}$ requires $\int_{\mathscr{S}_\rho} (\mathbf{K}^2 - \overset{\star}{\mathbf{K}}{}^2 \widehat{\mathbf{N}}^{-2}) \sqrt{\mathbf{d}}\, \overset{\circ}{\boldsymbol{\epsilon}} = -16\pi + \mathcal{O}(\rho^{-1})$

$\lim_{\rho \to \infty} m_H$ is finite $\Longrightarrow$ for all $j = 1, 2, \ldots, \mathcal{N}_j$

$$\widehat{\mathbf{N}}_{1,j}^{[log]} = \widehat{\mathbf{N}}_{2,j}^{[log]} = \mathbf{K}_{1,j}^{[log]} = 0$$
$$\mathbf{K}_{2,i}^{[log]} = \mathbf{K}_0 \left(2\,\mathbf{K}_{3,i}^{[log]} + \mathbf{K}_0^2\,\widehat{\mathbf{N}}_{4,i}^{[log]} \right) \cdot \left[\mathbf{a}_{(-1)}\,\mathbf{K}_0 + 4\,\mathbf{K}_1 \right]^{-1}$$
$$\widehat{\mathbf{N}}_{3,i}^{[log]} = -2 \left(2\,\mathbf{K}_{3,i}^{[log]} + \mathbf{K}_0^2\,\widehat{\mathbf{N}}_{4,i}^{[log]} \right) \cdot \left(\mathbf{K}_0 \left[\mathbf{a}_{(-1)}\,\mathbf{K}_0 + 4\,\mathbf{K}_1 \right] \right)^{-1}$$

The finiteness of the Bondi angular momentum: I.R.

- details of the construction in arXiv:2401.14251 [to appear in PRD]
- for any axial vector field ϕ^a

$$J[\phi] = -(8\,\pi)^{-1} \int_{\mathscr{S}_\rho} \left(\phi^a [\overline{q}_a \mathbf{k} + q_a \overline{\mathbf{k}}] \right) \sqrt{\mathbf{d}}\, \overset{\circ}{\boldsymbol{\epsilon}}$$

- since $\sqrt{\mathbf{d}} \sim \rho^2$ the Bondi angular momentum cannot be finite unless for all $j = 1, 2, \ldots, \mathcal{N}_j$
$$\mathbf{k}_{1,j}^{[log]} = \mathbf{k}_{2,j}^{[log]} = 0$$

Completion of the proof

- To obtain the desired restrictions, we substitute the updated form of the asymptotic expansions into the **parabolic-hyperbolic system** and sort the terms with respect to powers of ρ^{-1} and also of $\log \rho$. $\implies$

$$\widehat{\mathbf{N}}_{1,j}^{[log]} = \widehat{\mathbf{N}}_{2,j}^{[log]} = \widehat{\mathbf{N}}_{3,j}^{[log]} = \widehat{\mathbf{N}}_{4,j}^{[log]} = 0$$
$$\mathbf{k}_{1,j}^{[log]} = \mathbf{k}_{2,j}^{[log]} = \mathbf{k}_{3,j}^{[log]} = 0\,, \qquad \mathbf{K}_{1,j}^{[log]} = \mathbf{K}_{2,j}^{[log]} = \mathbf{K}_{3,j}^{[log]} = \mathbf{K}_{4,j}^{[log]} = 0$$

Inspecting the consequences

- The above findings suggest using the following **asymptotic expansions** in the parabolic-hyperbolic form of constraints **for constrained variables**:

  - $\widehat{\mathbf{N}} \longrightarrow \widehat{\mathbf{N}}_0 + \widehat{\mathbf{N}}_1\,\omega + \widehat{\mathbf{N}}_2\,\omega^2 + \widehat{\mathbf{N}}_3\,\omega^3 + \widehat{\mathbf{N}}_4\,\omega^4 + \omega^4 w_{\widehat{\mathbf{N}}}(\omega)$
  - $\mathbf{K} \longrightarrow \mathbf{K}_0 + \mathbf{K}_1\,\omega + \mathbf{K}_2\,\omega^2 + \mathbf{K}_3\,\omega^3 + \mathbf{K}_4\,\omega^4 + \omega^4 w_{\mathbf{K}}(\omega)$
  - $\mathbf{k} \longrightarrow \mathbf{k}_0 + \mathbf{k}_1\,\omega + \mathbf{k}_2\,\omega^2 + \mathbf{k}_3\,\omega^3 + \omega^3 w_{\mathbf{k}}(\omega)$

- The functions $w_{\widehat{\mathbf{N}}}(\omega), w_{\mathbf{K}}(\omega), w_{\mathbf{k}}(\omega)$, are of class $C^0\big([0,\omega_0), C^\infty(\mathbb{S}^2)\big)$ and **vanish at** $\partial\Sigma$, thus they can represent **all the higher-order log-terms** that may still occur.

- The coefficients $(\widehat{\mathbf{N}}_4, \mathbf{k}_2, \mathbf{K}_1)$ represent the **asymptotic degrees of freedom** since

  - All of the "**bold-faced coefficients in black**" can be **derived from the free data**, $(\mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$, and from the asymptotic degrees of freedom $(\widehat{\mathbf{N}}_4, \mathbf{k}_2, \mathbf{K}_1)$.

The asymptotic degrees of freedom and the asymptotic charges

- The terminology **asymptotic degrees of freedom** gets completely adequate by noticing that these quantities, $(\widehat{\mathbf{N}}_4, \mathbf{k}_2, \mathbf{K}_1)$, together with the **free data**, $(\mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$, determine:

- The **Bondi energy**

$$
m_B = \frac{1}{32\pi} \int_{\partial\Sigma} \left[\frac{\mathbf{K}_1{}^3}{\boldsymbol{\kappa}_0} + 8\,\widehat{\mathbf{N}}_4\,\boldsymbol{\kappa}_0{}^3 - 4\,\boldsymbol{\kappa}_0\,\boldsymbol{\kappa}_3 + 12\,\mathbf{a}_1\,\boldsymbol{\kappa}_0{}^2 - 2\,\mathbf{K}_1\left[\boldsymbol{\kappa}_0{}^{-1} + 3\,\mathbf{a}_0\,\boldsymbol{\kappa}_0 - 4\,\boldsymbol{\kappa}_2\right] \right.
$$
$$
- \left\{ \boldsymbol{\kappa}_0{}^5\,\mathbf{N}_1{}^{-1/2}\,\overline{\eth}\left(\mathbf{N}_1{}^{3/2}\boldsymbol{\kappa}_0{}^{-4}\mathbf{K}_1\right) + 2\,\boldsymbol{\kappa}_0\,\mathbf{N}_2{}^2\,\overline{\eth}\left(\mathbf{N}_2{}^{-1}\boldsymbol{\kappa}_0\right) + \tfrac{1}{4}\,\boldsymbol{\kappa}_0{}^5\,\eth\overline{\eth}\left(\boldsymbol{\kappa}_0{}^{-6}\,\mathbf{K}_1\right) \right.
$$
$$
\left. \left. - \tfrac{3}{2}\,\boldsymbol{\kappa}_0{}^{-2}\,\mathbf{K}_1\,\eth\overline{\eth}\,\boldsymbol{\kappa}_0 + \tfrac{1}{4}\,\boldsymbol{\kappa}_0\,\eth\overline{\eth}\,\mathbf{K}_1 + \text{``}cc\text{''} \right\} \right] \overset{\circ}{\boldsymbol{\epsilon}}
$$

- The components of the **Bondi angular momentum**

$$
J_B[\phi_{(i)}] = -(16\,\pi)^{-1} \int_{\partial\Sigma} [(\phi_{(i)}^a \overline{q}_a)\,\mathbf{k}_2 + (\phi_{(i)}^a q_a)\,\overline{\mathbf{k}}_2]\,\overset{\circ}{\boldsymbol{\epsilon}}
$$

  - where $\phi_{(i)}^a$, $(i = 1, 2, 3)$, denote the **axial Killing vector fields**

$$
\overset{\circ}{\phi}_1{}^a = -\sin\varphi\,(\partial_\vartheta)^a - \cot\vartheta\,\cos\varphi\,(\partial_\varphi)^a \quad \& \quad \overset{\circ}{\phi}_2{}^a = \cos\varphi\,(\partial_\vartheta)^a - \cot\vartheta\,\sin\varphi\,(\partial_\varphi)^a \quad \& \quad \overset{\circ}{\phi}_3{}^a = (\partial_\varphi)^a
$$

  of a **centre-of-mass unit sphere reference system** $[\int_{\partial\Sigma} \vec{x}\,\overset{\circ}{\boldsymbol{\epsilon}} = 0]$.

- The **algebraic conditions** on $\partial\Sigma$: $\boxed{\mathbf{a}_{(-1)} = const \ \& \ \overset{\circ}{\widehat{\mathbf{K}}}_{qq0} = \frac{1}{2}\,\boldsymbol{\kappa}_0 \cdot \eth\bar{\eth}\,\boldsymbol{\kappa}_0^{-2}}$ &

  - $\widehat{\mathbf{N}} \longrightarrow \widehat{\mathbf{N}}_0 + \widehat{\mathbf{N}}_1\,\omega + \widehat{\mathbf{N}}_2\,\omega^2 + \widehat{\mathbf{N}}_3\,\omega^3 + \widehat{\mathbf{N}}_4\,\omega^4 + \omega^4 w_{\widehat{\mathbf{N}}}(\omega)$
  - $\mathbf{K} \longrightarrow \mathbf{K}_0 + \mathbf{K}_1\,\omega + \mathbf{K}_2\,\omega^2 + \mathbf{K}_3\,\omega^3 + \mathbf{K}_4\,\omega^4 + \omega^4 w_{\mathbf{K}}(\omega)$
  - $\mathbf{k} \longrightarrow \mathbf{k}_0 + \mathbf{k}_1\,\omega + \mathbf{k}_2\,\omega^2 + \mathbf{k}_3\,\omega^3 + \omega^3 w_{\mathbf{k}}(\omega)$

- We get a **Fuchsian-type** (singular at $\omega = 0$) **equation** for the vector-valued variable $\underline{W} = (w_{\widehat{\mathbf{N}}}, w_{\mathbf{K}}, w_{\mathbf{k}})^T$, comprised of the residuals, where $p \in \mathscr{S}^2$ and $0 < \omega < \omega_0$:

$$\partial_\omega \underline{W}(\omega, p) = \frac{1}{\omega} diag(0, -3, -1)\,\underline{W}(\omega, p) \qquad (*)$$
$$+ \underline{H}\left(\omega, p; \widehat{\mathbf{N}}_4(p), \mathbf{k}_2(p), \mathbf{K}_1(p), \underline{W}(\omega, p), \eth\underline{W}, \bar{\eth}\underline{W}, \eth\bar{\eth}\underline{W}\right)$$

- where $\underline{H}$ is a (lengthy, but **explicitly known**) vector-valued function that is **smooth in each of its arguments**, and **regularly extends to** $\omega = 0$.

- A "formal solution" of $(*)$ can be given by this integral equation

$$\underline{W}(\omega, p) = diag\big[1, \omega^{-3}, \omega^{-1}\big] \times \int_0^\omega diag\big[1, s^3, s\big] \times \underline{H}(s, p)\,\mathrm{d}s \qquad (**)$$

- Since the integrand regularly extends to $s = 0$, we can perform the change of variables in the integral by replacing s with the product $\omega \cdot \tau$, which yields

$$\frac{1}{\omega}\,\underline{W}(\omega, p) = \int_0^1 diag\big[1, \tau^3, \tau\big] \times \underline{H}(\omega \cdot \tau, p)\,\mathrm{d}\tau \qquad (***)$$

Our second main result:

Theorem II.

- Choose **free data** $(\mathbf{N}, \mathbf{a}, \mathbf{b}, \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{qq})$ on Σ such that it **satisfies the asymptotically hyperboloidal falloff conditions** and **it is smooth on** $\Sigma \cup \partial\Sigma$.

- Suppose that $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are **smooth** [i.e., of class $C^{\infty}\big((0, \omega_0), C^{\infty}(\mathbb{S}^2)\big)$], solutions on Σ such that $\widehat{\mathbf{N}} > 0$ there.

- Then, the constrained fields $(\widehat{\mathbf{N}}, \mathbf{K}, \mathbf{k})$ are also of class $C^{\infty}\big([0, \omega_0), C^{\infty}(\mathbb{S}^2)\big)$ on the whole of $\Sigma \cup \partial\Sigma$, i.e., **no logarithmic singularities occur**, **if and only if** the asymptotically hyperboloidal initial data set under consideration **admits well-defined Bondi energy and angular momentum**, and, **in addition**, for the free data

$$\mathbf{a}_{(-1)} = const \quad \& \quad \overset{\circ}{\mathbf{K}}_{qq0} = \tfrac{1}{2}\,\boldsymbol{\kappa}_0 \cdot \eth\bar{\eth}\,\boldsymbol{\kappa}_0^{-2}$$

also hold on $\partial\Sigma$.

Summary:

- We proved that **the existence of well-defined Bondi energy and angular momentum**, together with some mild restrictions on the free data, implies that **the generic solutions of the parabolic-hyperbolic form of the constraint equations are free of logarithmic singularities**.

- We eliminated the restrictions imposed by Beyer and Ritchie on constrained fields. Additionally, we considerably weakened their restrictions on free data.

- Since Cauchy developments of smooth, asymptotically hyperboloidal initial data have smooth, conformal boundaries, **our result confirms** Penrose's **smoothness assumptions** used to define **asymptotically simple spacetimes**.

- Is there a way to **control the asymptotic charges** when constructing solutions to **the parabolic-hyperbolic form of the constraints**? [Work in progress.]

Thanks for your attention