Topologically non-trivial black holes and their spacetimes

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Plan of the talk

1 Topologically non-trivial horizons

- Isolated horizon structure
- Petrov Type D equation and its solutions

2 Embeddings of topologically non-trivial horizons

- Kerr-NUT-adS and spherical horizons
- Generalized Taub-NUT and higher genus horizons

3 Axis: regular and not

- Conical singularity: old and new
- Taub-NUT (incl. all Plebański-Demiański with NUT)

Topologically non-trivial horizons

On black hole horizons

- Boundary of a black hole: $\mathcal{B} = \mathcal{M} \setminus J^-(\mathcal{I}^+), \quad \mathcal{H} = \mathcal{H}^+ = \partial\mathcal{B}$
- Killing horizon: Take $\mathcal{L}_\ell g = 0$, then $\mathcal{H} = \mathcal{N}_\ell := \{g(\ell,\ell) = 0, \ell \neq 0\}$
  - Stationary BH event horizons are Killing
  - Null and non-expanding
  - Might not be a BH horizon (white, cosmological, neither...)
  - Still requires global symmetry
- Isolated Horizons
  Goal: capture horizon structure without a global symmetry.
  Isolated horizons analogues of black hole thermodynamics: 0th and 1st law Classical papers by Ashtekar, Lewandowski and many more from late 90' early 00'

Intrinsic structure of Isolated Horizons

Definition: *Non-expanding horizon* (H, g, ∇)

- $H =$ 3D null surface with g of signature $(0++)$ and topology of bundle $\Pi : H \to S$. S compact 2D surface with Riemannian ${}^{(2)}g$
  (generalisation of $H = S^2 \times \mathbb{R}$)
    - $g = \Pi^* \, {}^{(2)}g$.
- Non-expanding: $\ell \in \Gamma(TH)$ s.t. $g(\ell, \cdot) = 0 \implies \ell$ is Killing, $\mathcal{L}_\ell g = 0$
    - ℓ symmetry $\implies \ell' = f\ell$ also.
- H-null $\implies \nabla$ is not unique, instead given externally s.t. : ${}^{(2)}\nabla \; {}^{(2)}g = 0$

Definition: *Isolated Horizon* $(H, g, [\ell], \nabla)$

- "Stationary to the second order": $[\mathcal{L}_\ell, \nabla] = 0$
- Now $\ell \mapsto c_0\ell$, $c_0 \in \mathbb{R}$
- Rotation 1-form $\omega^{(\ell)}$: $\nabla\ell =: \omega^{(\ell)} \otimes \ell$, $\mathcal{L}_\ell\omega^{(\ell)} = 0$
    - Surface gravity: $\nabla_\ell \ell = \kappa^{(\ell)}\ell$, assume its constant on H
    - Pseudo-scalar invariant: ${}^{(2)}\mathrm{d}\omega^{(\ell)} = \Omega \; {}^{(2)}\eta$
- For IH: ω - unique, ℓ and κ up to a real const $\implies$ either $\underbrace{\text{extremal}}_{\kappa^{(\ell)}=0}$ or $\underbrace{\text{not}}_{\kappa^{(\ell)}\neq 0}$

Principle bundle structure

- So far: fibre bundle $\Pi : H \to S$
- More: principle bundle $G \hookrightarrow H \xrightarrow{\Pi} S$
  - G acts via the flow of ℓ
  - Connected group: $G = \mathbb{R}$ or $U(1)$. Classification by S and G.
- Bundle connection (choice of horizontal v.f. via ker $\tilde{\omega}$) $A = \tilde{\omega} \otimes \ell^*$ if
  - $\mathcal{L}_\ell \tilde{\omega} = 0$ ✓
  - $\tilde{\omega}(\ell) = 1 \implies \tilde{\omega} = \frac{\omega}{\kappa}$
- Bundle curvature $F = \frac{1}{\kappa} \mathrm{d}\omega \otimes \ell^*$
- $\tilde{\omega}$ and shorthand for principal bundle connection
- Characterisation of $U(1)$-bundles
  - $\int_S K^{(2)}\eta = 2\pi\chi_E(S) = 2 - 2$genus $\in \mathbb{N}$ - Euler characteristic
    K Gausian curvature of $(S, {}^{(2)}q)$
  - $\int_S \Omega^{(2)}\eta = 2\pi\chi_C(S) \in \mathbb{N}$ - Chern number

Petrov Type D Equation

- For embedded IH: Type D $\Leftrightarrow$ Weyl tensor is Type D at the horizon $C^{\alpha}{}_{\beta\gamma\delta}\,\big|_{\text{Horizon}}$
- For un-embedded IH: Type D $\Leftrightarrow K + \mathrm{i}\Omega - \frac{\Lambda}{3} \neq 0$
- Null co-frame on S: m_A $(A, B \in 1, 2)$
- Einstein Eqs $\big|_{\text{horizon}} \implies$ Petrov Type D eq: $\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(K + \mathrm{i}\Omega - \frac{\Lambda}{3}\right)^{-1/3} = 0$
- Known solutions: Λ-vacuum:
  - $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0 [1]
  - $H = S^2 \times \mathbb{R}$ with axial symmetry[2]
  - $H = S^3 \to S^2$, Hopf bundle with axial symmetry [3]
  - $H = S^3 \to S^2$, Hopf bundle with axial symmetry and conical singularities [4]
  - $H \to S$, with structure of $U(1)-$non -trivial bundle over S, smooth Riemann surface with genus > 0 [5]
- Spherical and non-axially symmetric solution - open problem!

[1] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[2] Phys. Rev. D 98 (2018), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski
[3] Phys. Rev. D 100 (2019), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz
[4] Phys. Rev. D 108 (2023), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and MO
[5] Phys. Rev. D 110 (2024), 024071, Jerzy Lewandowski, and MO

Spherical horizons

- For axial symmetry Type D Eq reduces to ODE. Full classification is known.
- Topology $H \xrightarrow{\Pi} S^2$, i.e. Hopf bundles and higher.
- Non-rotating: $d\omega = 0 \implies \Omega =$ const & Taub-NUT-...
- Rotating: $d\omega \neq 0 \implies \Omega = \Omega(x)$, 3 parameters & Kerr-NUT-...
- "Transversal" horizons: KVF other (!) than ℓ generates $U(1)$ symmetry, 5 parameters & Kerr-NUT-...
  - Two conically singular halves glued into a smooth (but topologically non-trivial) horizon, or
  - Topologically trivial horizon, but with conical singularities
- Extremal horizons with transversal structure: still topologically non-trivial

Higher genus horizon

- S - smooth, oriented, compact, Riemann surface $\implies$ topologically characterized only by genus
- $K = \frac{4\pi(1-\text{genus})}{\text{Area}}$ =const, Ω =const ($\Omega = 0$ for trivial horizons) $\implies$ non=rotating
- Toroidal case: explicit coordinates for all flat ($K = 0$) metrics
- Genus > 1: no easy coordinates, general arguments
  - Constant, negative curvature
  - $\text{Prin}_{U(1)}(S) \cong H^2(S;\mathbb{Z}) \cong \mathbb{Z}$ ($\sim$ first Chern class)
- ${}^{(2)}\omega$ defined up to
  - $U(1)$ gauge
  - 1-forms $\alpha_1, \ldots, \alpha_{2g}$ generating the first de Rham cohomology group $H^1(S, \mathbb{R}) \cong \mathbb{R}^{2\text{genus}}$ for any genus

Embeddings of topologically non-trivial horizons

Plebański-Demiański spacetimes

- Most general Petrov Type D solution - Weyl tensor has 2 double principle directions
- Lambda-electro-vaccum solution to EEs in 4D, with EM field aligned with principal null directions
- Generalisation of Schwarzschild, Kerr etc ...
- Parameters:

$$\underbrace{M}_{\text{"mass"}}, \underbrace{a}_{\text{Kerr}}, \underbrace{\alpha}_{\text{acceleration}}, \underbrace{e, \ g}_{\text{e.m. charges}},$$

$$\underbrace{\Lambda}_{\text{cosmological constant}}, \underbrace{l}_{\text{N(ewmn)-U(unti)-T(amburino)}}$$

,

- No topology restriction. Useful to distinguish 2D surfaces with curvature

$$\epsilon > \text{ or } = \text{ or } < 0$$

- 2 commuting Killing Vector fields $\approx$ time translation, rotation symmetry

Accelerated Kerr-NUT-adS and spherical troubles

$$\mathrm{d}s^2 = -\frac{1}{F^2}\left[\frac{\mathcal{Q}}{\Sigma}(\mathrm{d}t - A\mathrm{d}\phi)^2 + \frac{\Sigma}{\mathcal{Q}}\mathrm{d}r^2 + \frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma}\sin^2\theta(a\mathrm{d}t - \rho\mathrm{d}\phi)^2\right]$$

$\mathcal{Q} = \mathcal{Q}(r)$ – zeros define Killing Horizon,

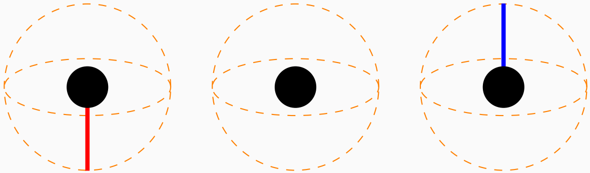
$$\mathcal{P} = \mathcal{P}(\theta) > 0, \quad \Sigma = \Sigma(r,\theta) \neq> 0, \quad \rho = \rho(r), \quad F = F(r,\theta) \neq 0$$

$$A = a\sin^2\theta - 2l\left(\cos\theta - 1\right), \quad A(\theta = 0) = 0, \quad A(\theta = \pi) = 4l \neq 0$$

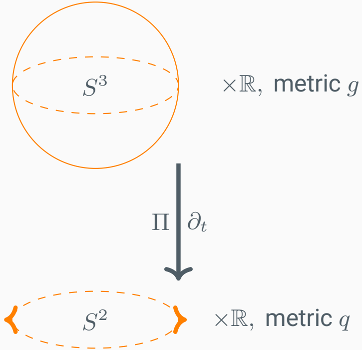
$\omega := \mathrm{d}t - A\mathrm{d}\phi$ not continuos at $\theta = \pi \implies$ singular half-axis.

Misner interpretation

$t' := t - 4l\phi \implies \omega := \mathrm{d}t - A'\mathrm{d}\phi, \quad A' = a\sin^2\theta - 2l\left(\cos\theta + 1\right)$ t is cyclic with period $8\pi l$! $\partial_t, \partial_{\phi_\pi}$ - cyclic



Orbits of Killing vector fields and the bundle structure

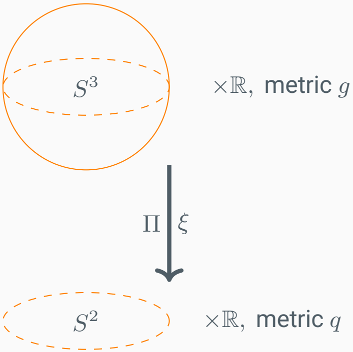


$U(1)$-principle bundle

- Base space: $S^2 \times \mathbb{R}$
- Fibres: group $U(1)$
- Total space: $S^3 \times \mathbb{R} \overset{\text{locally}}{\approx} S^2 \times U(1) \times \mathbb{R}$
- $U(1)$ preserves fibres and acts on them:
  - Freely: no non-trivial fixed points
  - Transitively: every point is reachable from any other

$$g = -\underbrace{(\Pi^* f)}_{\text{lapse function}} \omega \otimes \underbrace{\omega}_{\text{connection}} + \Pi^* \underbrace{q}_{\text{orbit-space metric}}, \quad (\partial_t)^\mu (\partial_t)_\mu = -(\Pi^* f)$$

Residual conical singularity on the space of the orbits!



Non-singular spacetimes

- Generically: $\xi = \partial_t + b\partial_\phi, \quad b \in \mathbb{R}$
- Exactly 2 (equivalent) choices of ξ with no conical singularity

$$\mathcal{P}(0) = \frac{\mathcal{P}(\pi)}{|(1-4bl)|}$$

- Generates horizon if $\xi = \ell = \partial_t + \frac{a}{\rho(r_H)\partial_\phi}$ & constraint on parameters.
- $\omega := \frac{g(\xi,\cdot)}{g(\xi,\xi)}$

$$g = -\underbrace{(\Pi^* f)}_{\text{lapse function}}\,\omega \otimes \underbrace{\omega}_{\text{connection}} + \Pi^* \underbrace{q}_{\text{orbit-space metric}}, \quad \xi^\mu \xi_\mu = -(\Pi^* f)$$

Every component is well defined $\implies$ the spacetime is non-singular, including horizon as a 3D manifold.
Generically ($\xi \neq \ell$) space of the null orbits is not smooth - quotient by two different vector fields.

Embedding spacetimes

Generalised Taub-NUT-(anti-) de Sitter (M, l, Λ) with t cyclic.

$$g = -f(r)\left(\mathrm{d}t + l\frac{i(\zeta\mathrm{d}\bar{\zeta} - \bar{\zeta}\mathrm{d}\zeta)}{1+\frac{1}{2}\epsilon\zeta\bar{\zeta}}\right)^2 + f(r)^{-1}\mathrm{d}r^2 + (r^2+l^2)\frac{2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{(1+\frac{1}{2}\epsilon\zeta\bar{\zeta})^2},$$

$$f(r) = \frac{\epsilon(r^2-l^2) - 2Mr - \Lambda(\frac{1}{3}r^4 + 2l^2r^2 - l^4)}{r^2+l^2}, \quad \epsilon = 0, \pm 1$$

- Spherical: $\epsilon = 1$, Taub-NUT-adS $\zeta = \sqrt{2}\tan\frac{1}{2}\theta\exp(\mathrm{i}\phi)$
    - $l = 0$ Schwarzschild-(anti-) de Sitter
- Planar: $\epsilon = 0$, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
- Hyperbolic: $\epsilon = -1$, S with genus > 1, $S = \mathbb{H}/\Gamma$, $\Gamma \subset$ PSL$(2, \mathbb{R})$
- Bundle topology: $U(1) \hookrightarrow P \times \mathbb{R} \xrightarrow{\Pi} S \times \mathbb{R}$.
- Connection defined up to $H^1(S, \mathbb{R}) \implies$ Gravitational Aharonov-Bohm

Corollary:

Conical singularity in NUT is observer-dependent

based on: arXiv:2507.21238, Ivan Kolář, Pavel Krtouš, MO

Conical singularity is observer-dependent

$\boldsymbol{c}$ - cyclic Killing vector field. Defines τ and has closed orbits.
Axis $= \{x \in M : \boldsymbol{c} = 0\}$.

1. The elementary flatness condition - assures $2\pi-$periodicity.

$$\frac{\nabla_\alpha(\boldsymbol{c}^2)\nabla^\alpha(\boldsymbol{c}^2)}{4\boldsymbol{c}^2} \xrightarrow{\text{axis}} 1 \implies \lim_{||\boldsymbol{c}||\to 0^+} \left|\left|\mathbf{d}||\boldsymbol{c}||\right|\right| = 1$$

Assumptions:

- In some region of (M, g) - stationary axially symmetric ST
- with (at least) a 2-dimensional Abelian algebra of KVFs - Γ
- such that there exist cyclic KVF $\boldsymbol{c} \in \Gamma$ with 2π periodic orbits.

Therefore $\exists$ infinitely many timelike KVFs ($\sim$ observers), choose any of them: $\boldsymbol{t} \in \Gamma$.
Assume that the singular axis point x is not a regular boundary point (otherwise extend the spacetime) and that x corresponds to a quasi regular-singularity.
"Where" is the singular axis? We need a notion of a boundary corresponding to a singular point.
Singular axis point $x \in \partial M$ and the axial Killing vector field $\boldsymbol{a} \in \Gamma$

$$||\boldsymbol{a}|| \to 0^+ \text{ , towards } x \text{ and } \lim_{||\boldsymbol{a}||\to 0^+} \left|\left|\mathbf{d}||\boldsymbol{a}||\right|\right| = 1 \, ,$$

Conical singularity

What is conical singularity? Wrong choice of the angular coordinates - ϕ is not 2π periodic.
Quasi-regular point - curvature in all parallelly propagated frames is regular.
Its measure: conicity On a 2D surface: take a circle centred at the axis, of length $L_\circ$ and (geodesic) radius $\rho_\circ$:

$$\mathcal{C}_{2\mathrm{D}} = \lim_{\rho_\circ \to 0+} \frac{L_\circ}{2\pi\rho_\circ} \; .$$

For regular axis: $\mathcal{C} = 1$.
In 4D spacetime: choose a 2D surface $\mathcal{S}$ and apply the above.
Usually $(t, r, \theta, \phi) \to (\theta, \phi)$.

- Does $\mathcal{C}_\mathcal{S}$ depend on the choice of $\mathcal{S}$?
- To which point of the axis is the limit calculated? Does the limit depend on the curve taken?
- The axis is not part of the manifold: how to distinguish points?

No better definition?

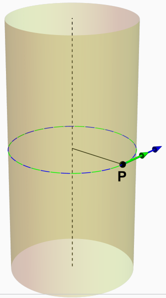
The axial KVF $\boldsymbol{a} \in \Gamma$ hence:

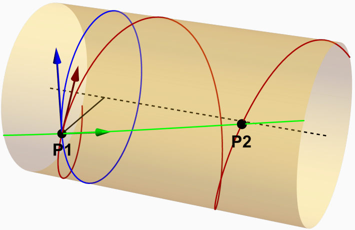
$$\boldsymbol{a} = \frac{1}{\mathcal{C}} \left(\boldsymbol{c} + \mathcal{T} \boldsymbol{t} \right) ,$$

for some

- $\mathcal{C}$ - conicity
- $\mathcal{T}$ - time-shift

The orbit of Γ. Left $\mathcal{T} = 0$, right $\mathcal{T} \neq 0$:





Transformation laws

Take a different observer (timelike KVF): $\tilde{\boldsymbol{t}} = \alpha \boldsymbol{t} + \beta \boldsymbol{c}, \alpha \neq 0$ then:

$$\boldsymbol{a} = \frac{1}{\mathcal{C}}\left(\boldsymbol{c} + \mathcal{T}\frac{\tilde{\boldsymbol{t}} - \beta \boldsymbol{c}}{\alpha}\right) = \frac{\alpha - \beta\mathcal{T}}{\alpha\mathcal{C}}\left(\boldsymbol{c} + \frac{\mathcal{T}}{\alpha - \beta\mathcal{T}}\tilde{\boldsymbol{t}}\right) \ ,$$

The conicity and time-shift transform as:

$$\mathcal{C}(\tilde{\boldsymbol{t}}) = \frac{\mathcal{C}(\boldsymbol{t})}{\left|1 - \frac{\beta}{\alpha}\mathcal{T}(\boldsymbol{t})\right|} \ , \quad \mathcal{T}(\tilde{\boldsymbol{t}}) = \frac{\mathcal{T}(\boldsymbol{t})}{\alpha - \beta\mathcal{T}(\boldsymbol{t})}$$

- In general both are observer-dependent!
- If $\mathcal{T}(\boldsymbol{t}) = 0$ for some $\boldsymbol{t}$ then $\forall \tilde{\boldsymbol{t}}$ we have $\mathcal{T}(\tilde{\boldsymbol{t}}) = 0$ and $\mathcal{C}(\boldsymbol{t}) = \mathcal{C}(\tilde{\boldsymbol{t}})$.
  - The observer-dependency is caused by the time-shift, which is caused by NUT
- Observers without conicity:

$$\frac{\boldsymbol{t}_{I/II}}{\alpha} = \boldsymbol{t} + \frac{1 \pm \mathcal{C}}{\mathcal{T}}\boldsymbol{c} \ ,$$

- If we have two parts of the axis: $(+) \equiv (\theta = 0), \ (-) \equiv (\theta = \pi)$ there are observers without conicity difference $(\Delta\mathcal{C} := \frac{\mathcal{C}_+ - \mathcal{C}_-}{\mathcal{C}_+ + \mathcal{C}_-})$

$$\frac{\boldsymbol{t}_{1/2}}{\alpha} = \boldsymbol{t} + \frac{\mathcal{C}_+ \pm \mathcal{C}_-}{\mathcal{C}_+\mathcal{T}_- \pm \mathcal{C}_-\mathcal{T}_+}\boldsymbol{c}$$

Examples: Taub-NUT

Taub-NUT (l) spacetime without periodic time identification:

$$\boldsymbol{g} = -f(r)(\mathbf{d}t + 2l(\cos\theta + s)\mathbf{d}\phi)^2 + \frac{\mathbf{d}r^2}{f(r)} + (l^2 + r^2)(\mathbf{d}\theta^2 + \sin^2\theta\mathbf{d}\phi^2) \ .$$

$$f(r) := \frac{r^2 - 2mr - l^2}{r^2 + l^2} \ .$$

Two axial KVFs and two axis parts

$$\boldsymbol{a}_\pm = (\partial_\phi + 2l(s \pm 1)\partial_t)$$

Consider an observer $\boldsymbol{t} = t^t\boldsymbol{\partial}_t + t^\phi\boldsymbol{\partial}_\phi$, then always possible to regularise a part:

$$\mathcal{C}_\pm(\boldsymbol{t}) = \frac{1}{\left|1 + 2l(s \pm 1)\frac{t^\phi}{t^t}\right|} \ , \quad \mathcal{T}_\pm(\boldsymbol{t}) = -\frac{2l(s \pm 1)\frac{1}{t^t}}{1 + 2l(s \pm 1)\frac{t^\phi}{t^t}} \ ,$$

$$\Delta\mathcal{C} = \frac{2}{1 + \left|\frac{1+2(s+1)l\frac{t^\phi}{t^t}}{1+2(s-1)l\frac{t^\phi}{t^t}}\right|} - 1 \ .$$

Similarly for accelerated Kerr-NUT-(anti-) de Sitter.
Reproduces conditions for non-singular interpretation a'la Misner.

Thank you for your attention!