General covariance and effective spherically symmetric models

Juan Carlos del Águila and Hugo A. Morales-Técotl

Departamento de Física, UAM Iztapalapa, Ciudad de México, México

Abstract.

Motivated by models obtained in spherically symmetric Loop Quantum Gravity, a criterion for testing their general covariance is put forward that adapts invariance under general spacetime diffeomorphisms of the Einstein-Hilbert action to the case of effective canonical models. Inverse triad and holonomy corrections are considered as examples. It turns out further quantum corrections of the metric are required for general covariance to hold.

I Introduction.

- Investigating the nature of spacetime lies at the heart of any quantum gravity theory.
- Loop Quantum Gravity (LQG) in its Hamiltonian form replaces the canonical algebra of Geometrodynamcs by a flux-holonomy one possessing a discrete representation in an adequate Hilbert space [1]. For the covariant form see eg [2].
- Symmetrical models turn out useful to investigate LQG: The symmetric sector of the classical theory is subject to the quantization procedure [3]. However, a complete quantum symmetry reduction is yet to be completed [4].
- Effective theories represented in classical phase space, can be defined through semiclassical states either in Hamiltonian or path integral form [5], thus allowing to contrast their behaviors directly.
- Avoidance of classical singularities result within effective theories as bounded quantities like, say, curvature invariants, expansion and shear [6-7].
- Remarkably, general covariance of effective models turns out not to be automatic [8-9] and assessing its fulfillment is crucial as has been considered recently by Belfaqih-Bojowald-Brahma-Duque [10], and Zhang-Lewandowski-Ma-Yang [11].
- Here we describe a proposal put forward in [12] that considers general covariance of General Relativity as the starting point for probing effective spherically symmetric models. It yields, in line with [10-11], that a the spacetime metric must be subject to further quantum corrections.

II Spherically symmetric models

Spherically symmetric models using Ashtekar-Barbero variables can be described by

$$ds^2 = -N^2dt^2 + \frac{(E^\varphi)^2}{E^r}\left(dr + N_r dt\right)^2 + E^r d\Omega^2 \tag{1}$$

$$\{K_r(t,r), E^r(t,r')\} = 2\gamma\delta(r,r'), \quad \{K_\varphi(t,r), E^\varphi(t,r')\} = 2\gamma\delta(r,r') \tag{2}$$

subject to the constraints

$$C_{\text{class}}[N_r] = \int dr N_r \left[(E^r)' K_r - E^\varphi K_\varphi'\right], \tag{3}$$

$$H_{\text{class}}[N] = -\int dr N \left[\frac{K_\varphi}{2\gamma^2\sqrt{|E^r|}}\left(4K_r E^r + K_\varphi E^\varphi\right) + \frac{2E^\varphi}{\sqrt{|E^r|}}(1-\Gamma_\varphi^2) + 4\sqrt{|E^r|}\,\Gamma_\varphi'\right]$$

with $\Gamma_\varphi = -E^{r'}/2E^\varphi$ and which fulfill the constraint algebra

$$\left\{H_c[N], C_c[N_r]\right\} = H_c[N_r N'], \left\{C_c[N_r], C_c[M_r]\right\} = -C_c[M_r N_r' - N_r M_r'], \tag{4}$$

$$\left\{H_c[N], H_c[M]\right\} = C_c\left[\frac{E^r}{(E^\varphi)^2}(MN' - NM')\right]. \tag{5}$$

III General Covariance from GR to effective spherical [11]

- Classical covariance. For the Einstein-Hilbert action subject to a diffeomorphism $\phi: M \to M, \delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu}$,

$$S_G[g_{\mu\nu}] = \frac{1}{8\pi}\int_M \sqrt{-g}R\,d^4x, \delta_\xi S_G = -\frac{1}{8\pi}\int_M \sqrt{-g}\xi_\nu \nabla_\mu G^{\mu\nu} d^4x, \tag{6}$$

invariance yields the contracted Bianchi identity

$$\delta_\xi S_G = 0 \Rightarrow \nabla_\mu G^{\mu\nu} = 0. \tag{7}$$

- Covariance of the effective theory

$$S_G \to S^{\text{eff}}, \delta_\xi S^{\text{eff}} = 0 \Rightarrow \nabla_\mu \mathscr{G}^{\mu\nu} = 0, \tag{8}$$

here $\mathscr{G}^{\mu\nu}$ is an effective Einstein tensor

$$\mathscr{G}^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S^{\text{eff}}}{\delta g_{\mu\nu}}. \tag{9}$$

- Relation with the Hamiltonian form of effective theory. We have

$$\vec{\mathscr{G}} = \frac{1}{\sqrt{-g}}\mathbf{g}_\Gamma^{-1}\vec{S}_\Gamma, \quad \vec{\mathscr{G}} = \begin{bmatrix}\mathscr{G}^{00}\\ \mathscr{G}^{01}\\ \vdots \\ \mathscr{G}^{23}\\ \mathscr{G}^{33}\end{bmatrix}, \tag{10}$$

equivalent to a rewriting of

$$\frac{\delta S^{\text{eff}}}{\delta \vec{\Gamma}_P} = \frac{\delta S^{\text{eff}}}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta \vec{\Gamma}_P}, \ \vec{S}_\Gamma = \frac{\delta S^{\text{eff}}}{\delta \vec{\Gamma}_P}, \ \vec{\Gamma}_P = (N, N^a, P^A), \tag{11}$$

so one can check (8).

- **Spherical symmetry.**

Classical model

$$\nabla_\mu \mathscr{G}^{\mu\nu} = \sum_{i=0}^{2}\left[A^\nu_{(i)}\partial_r^i\left(\dot{E}^r - \{E^r, H_T[N,N_r]\}\right) + B^\nu_{(i)}\partial_r^i\left(\dot{E}^\varphi - \{E^\varphi, H_T[N,N_r]\}\right)\right]$$

$$\nabla_\mu \mathscr{G}^{\mu\nu}|_{\text{eom}} = 0 \quad \text{Classical} \tag{12}$$

Effective model (triad corrections)

$$\frac{1}{\sqrt{E^r}} \to \frac{\alpha_1(E^r)}{\sqrt{E^r}} \quad \text{and} \quad \sqrt{E^r} \to \alpha_2(E^r)\sqrt{E^r} \tag{13}$$

$$\nabla_\mu \mathscr{G}^{\mu\nu} = \sum_{i=0}^{2}\left[\bar{A}^\nu_{(i)}\partial_r^i\left(\dot{E}^r - \{E^r, H_T^{eff}[N,N_r]\}\right) + \bar{B}^\nu_{(i)}\partial_r^i\left(\dot{E}^\varphi - \{E^\varphi, H_T^{eff}[N,N_r]\}\right)\right]$$

$$+(\alpha_2^2 - 1)f^\nu_{(1)} + \frac{d\alpha_2}{dE^r}f^\nu_{(2)}. \tag{14}$$

$\alpha_2 = 1$ reduces to the classical case. However, defining an effective metric

$$\bar{E}^r = \alpha_2^2 E^r, E^\varphi \to E^\varphi,$$

$$d\bar{s}^2_{(1)} = \bar{g}^{(1)}_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \frac{(E^\varphi)^2}{\bar{E}^r}\left(dr + N_r dt\right)^2 + \bar{E}^r d\Omega^2, \tag{15}$$

and repeating the procedure yields

$$\nabla_\mu \mathscr{G}^{\mu\nu} = \sum_{i=0}^{2}\left[\mathscr{A}^\nu_{(i)}\partial_r^i\left(\dot{E}^r - \{E^r, H_T^{eff}[N,N_r]\}\right) + \mathscr{B}^\nu_{(i)}\partial_r^i\left(\dot{E}^\varphi - \{E^\varphi, H_T^{eff}[N,N_r]\}\right)\right]$$

and hence

$$\nabla_\mu \mathscr{G}^{\mu\nu}|_{\text{eff-eom}} = 0. \quad \text{Effective} \tag{16}$$

IV Discussion

- General covariance of effective spherically symmetric models has been shown to descend from that of the full theory at the level of the action [12], but corrections for the metric are required. This in accordance with previous approaches [10-11].
- Here we have presented inverse triad corrections in the spherically symmetric cases but similar results hold for holonomy corrections [10-12].
- Within spherical symmetry, there seems to be a connection between our approach and previous ones as indicated for the examples that have been studied in common yielding same results. However, there are other examples not included in one or other of the approaches. To unveil this requires more work.
- Other symmetries than spherical would be of interest to check in the different approaches.
- As for our approach it may be enlightening to study the connection between the manifestly covariant formalism and the canonical one along the lines of [13].

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Bibliography

[1] Thiemann T Modern 2007 Canonical Quantum General Relativity Cambridge University Press. Rovelli C 2004 Quantum Gravity. Cambridge University Press. Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report *Class. Quantum Grav.* 21 R53

[2] Rovelli C and Vidotto F 2015 Covariant Loop Quantum Gravity Cambridge University Press.

[3] Ashtekar A and Bojowald M 2006 Quantum geometry and the Schwarzschild singularity *Class.Quantum Grav.* 23 391

[4] Beetle C Engle J S, Hogan M E and Mendonça P 2017 Diffeomorphism invariant cosmological sector in loop quantum gravity *Class. Quantum Grav.* 34 225009

[5] Ashtekar A, Campiglia M and Henderson A 2010 Path integrals and the WKB approximation in Loop Quantum Cosmology *Phys. Rev. D* 82 124043

[6] Corichi A and Singh P 2006 Loop quantization of the Schwarzschild interior revisited Class. Quantum Grav. 33 055006

[7] Ashtekar A, Olmedo J and Singh P 2018 Quantum transfiguration of Kruskal black holes Phys. Rev. Lett. 121 241301

[8] Bojowald M 2020 No-go result for covariance in models of loop quantum gravity *Phys.Rev.D* 102 046006

[9] Gambini R, Olmedo J and Pullin j 2022 Towards a quantum notion of covariance in spherically symmetric loop quantum gravity *Phys.Rev.D* 105 026017

[10] Belfaquih I H, Bojowald M, Brahma S and Duque E I 2025 Lessons for loop quantum gravity from emergent modified gravity *Phys. Rev. D* 111 086027

[11] Zhang C, Lewandowski J, Ma Y and Yang J 2025 J Black holes and covariance in effective quantum gravity *Phys. Rev. D* 111 L081504

[12] del Águila J C and Morales-Técotl H A 2025 Testing general covariance in effective models motivated by loop quantum gravity *Class. Quantum Grav.* 42 105002

[13] Lee J and Wald R M Local 1990 Symmetries and constraints *Jour. Math. Phys.* 31 725