Codifferential calculus

Patryk Mieszkalski ♠
University of Wrocław
♠patryk.mieszkalski@uwr.edu.pl

Introduction

Noncommutative Differential Geometry (NCDG) has attracted a lot of attention in last decades. Woronowicz Hopf algebra bicovariant differential calculi provided a new impetus for this development [4]. In particular, they become very popular as a tool for investigations of Quantum Groups (QG).

Similarly, the formal dual of differential calculus over algebras generates codifferential calculus over coalgebras, introduced by Doi in [2]. There is no vast literature on this subject.

Here, we are mainly interested in using first-order codifferential calculuson Hopf algebra as a tool for finding covariant first-order differential calculus on module-algebras. We will show the procedure on examples of κ-Poincaré Hopf algebra and κ-Minkowski spacetime algebra.

1 Main definitions

Definition: Differential calculus

The pair (Υ, δ) we call coderivation over coalgebra H if $\Upsilon \in^H \mathfrak{M}^H$ is bicomodule, $\delta: \Upsilon \to H$ is a linear map and coLiebnitz rule is fulfilled:

$$\Delta \circ \delta = (\mathrm{id} \otimes \delta) \circ \Delta_L + (\delta \otimes \mathrm{id}) \circ \Delta_R\,.$$

If the kernel of a right comodule morphism $\sigma_R := (\delta \otimes \mathrm{id}) \circ \Delta_R$ is trivial, we call (Υ, δ) a first-order codifferential calculus.

If H is Hopf algebra, Υ is Hopf bimodule (Woronowicz's bicovariant bimodule) and:

$$\delta(a \triangleright \upsilon \triangleleft b) = a\delta(\upsilon)b,$$

then we call (Υ, δ) bicovariant first-order codifferential calculus.

Definition: Covariant differential calculus

Let H be Hopf algebra and $\mathcal{A}$ be module algebra with action $\triangleright: H \otimes \mathcal{A} \to \mathcal{A}$.

We call differential calculus $(\Omega_{\mathcal{A}}, \mathrm{d})$ over algebra $\mathcal{A}$ covariant with respect to H action there is an action $\blacktriangleright: H \otimes \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}$ and:

$$a \blacktriangleright \mathrm{d}f = \mathrm{d}(a \triangleright f).$$

Let (Υ_H, δ) be bicovariant codifferential calculus over H. We call $\Omega_{\mathcal{A}}$ covariant with respect to Υ_H if there exist pairing $\lrcorner: \Upsilon_H \otimes \Omega_{\mathcal{A}} \to \mathcal{A}$ and:

$$\upsilon \lrcorner \mathrm{d}f = \delta\upsilon \triangleright f,$$
$$\upsilon \lrcorner (a \blacktriangleright \omega) = (\upsilon a) \lrcorner \omega.$$

2 κ-Poincaré Hopf algebra

Let's introduce κ-Poincaré Hopf algebra by a system of generators $H = \mathrm{gen}\{\Pi_0,\ \Pi_0^{-1},\ P_j,\ N_j,\ M_j | j = 1, 2, 3\}$ (look [1])

$$\begin{array}{lll} [\mathcal{P}_i, \Pi_0] = 0, & [\mathcal{P}_j, \mathcal{P}_k] = 0, & [M_j, M_k] = i\epsilon_{jkl}M_l, \\ [M_j, \Pi_0] = 0, & [M_j, \mathcal{P}_k] = i\epsilon_{jkl}\mathcal{P}_l, & [N_j, M_k] = i\epsilon_{jkl}N_l, \\ [N_j, \Pi_0] = \frac{i}{\kappa}\mathcal{P}_j, & [N_j, \mathcal{P}_k] = -i\delta_{jk}\mathcal{P}_0, & [N_j, N_k] = -i\epsilon_{jkl}M_l. \end{array}$$

$$\begin{aligned} \Delta\Pi_0 &= \Pi_0 \otimes \Pi_0, \\ \Delta\mathcal{P}_j &= \mathcal{P}_j \otimes \Pi_0 + 1 \otimes \mathcal{P}_j, \\ \Delta M_j &= M_j \otimes 1 + 1 \otimes M_j, \\ \Delta N_j &= N_j \otimes 1 + \Pi_0^{-1} \otimes N_j - \frac{1}{\kappa}\epsilon_{jkl}\mathcal{P}_k\Pi_0^{-1} \otimes M_l. \end{aligned}$$

$$\mathcal{P}_0 = \frac{\kappa}{2}\left(\Pi_0 - \Pi_0^{-1}(1 - \frac{1}{\kappa^2}\overrightarrow{\mathcal{P}}^2)\right).$$

Lowest dimensional bicovariant first-order codifferential calculus

It is 5-dimensional with its right-free module basis: $\Upsilon_H = \langle \upsilon_C, \upsilon_0, \upsilon_j \rangle$.

$$\delta(\upsilon_0) = \Pi_0 - 1, \quad \delta(\upsilon_j) = \mathcal{P}_j, \quad \delta(\upsilon_C) = \kappa^2(\Pi_0 + \Pi_0^{-1} - 2) - \overrightarrow{\mathcal{P}}^2\Pi_0^{-1}.$$

$$\Delta_L(\upsilon_C) = \Pi_0^{-1} \otimes \upsilon_C + 2(\kappa\mathcal{P}_0 - \overrightarrow{\mathcal{P}}^2\Pi_0^{-1}) \otimes \upsilon_0 - 2\mathcal{P}_k\Pi_0^{-1} \otimes \upsilon_k,$$
$$\Delta_L(\upsilon_0) = \Pi_0 \otimes \upsilon_0, \qquad \Delta_L(\upsilon_j) = \mathcal{P}_j \otimes \upsilon_0 + 1 \otimes \upsilon_j.$$

$$[N_j, \upsilon_0] = -\frac{i}{\kappa}\upsilon_j, \qquad [M_j, \upsilon_k] = i\epsilon_{jkl}\upsilon_l,$$
$$[N_j, \upsilon_k] = i\delta_{jk}(\frac{1}{2\kappa}\upsilon_C - \kappa\upsilon_0).$$

2.1 κ-Minkowski

Lowest dimensional covariant first-order differential calculus

It is 5-dimensional with its right-free module basis $\Omega_{\mathcal{A}} = \left\langle \theta^C, \theta^0, \theta^j \right\rangle$ dual to basis of Υ_H.

The partial derivatives can be found:

$$\partial_X^\theta(f) = \delta(\upsilon_X) \triangleright f.$$

can be found from:

$$\upsilon_X \lrcorner f\theta^Y = (\upsilon_{X(-1)} \triangleright f)(\upsilon_{X<0>} \blacktriangleright \theta^Y) = (\upsilon_X \lrcorner \theta^Z)A_Z^{\ Y}(f).$$

$$\begin{aligned} f\theta^0 &= \theta^0(\Pi_0 f) + \theta^i(\mathcal{P}_i f) + 2\theta^C[(\kappa\mathcal{P}_0 - \overrightarrow{\mathcal{P}}^2\Pi_0^{-1})f], \\ f\theta^j &= \theta^j f - 2\theta^C(\mathcal{P}_j\Pi_0^{-1}f), \\ f\theta^C &= \theta^C(\Pi_0^{-1}f). \end{aligned}$$

$$\Pi_0 \blacktriangleright_H \theta^X = 1, \quad M_j \blacktriangleright_H \theta^k = i\epsilon_{jkl}\theta^l,$$
$$N_j \blacktriangleright_H \theta^C = -\frac{i}{2\kappa}\theta^j, \quad N_j \blacktriangleright_H \theta^0 = i\kappa\theta^j, \quad N_j \blacktriangleright_H \theta^k = \frac{i}{\kappa}\delta_{jk}\theta^0.$$

Comparison with literature [3]

Literature	My result
$[x^0, \mathrm{d}x^0] = \frac{1}{\kappa^2}\phi,$	$[x^0, \mathrm{d}x^0] = \frac{1}{\kappa}\mathrm{d}x^0 + 2\theta^C,$
$[x^j, \mathrm{d}x^0] = \frac{1}{\kappa}\mathrm{d}x^j,$	$[x^j, \mathrm{d}x^0] = \frac{1}{\kappa}\mathrm{d}x^j,$
$[x^0, \mathrm{d}x^j] = 0,$	$[x^0, \mathrm{d}x^j] = 0,$
$[x^j, \mathrm{d}x^k] = \delta_{jk}\frac{1}{\kappa}(\mathrm{d}x^0 - \frac{1}{\kappa}\phi),$	$[x^j, \mathrm{d}x^k] = -2\delta_{jk}\theta^C,$
$[x^0, \phi] = \mathrm{d}x^0,$	$[x^0, \theta^C] = -\frac{1}{\kappa}\theta^C,$
$[x^j, \phi] = \mathrm{d}x^j,$	$[x^j, \theta^C] = 0,$

There is an agreement:

$$\phi = \kappa\mathrm{d}x^0 + 2\kappa^2\theta^C.$$

References

[1] A Borowiec and A Pachol. Classical basis for kappa-poincare algebra and doubly special relativity theories. 2009.

[2] Yukio Doi. Homological coalgebra. *J. Math. Soc. Japan*, 33(1):31–50, January 1981.

[3] Andrzej Sitarz. Noncommutative differential calculus on the kappa-minkowski space. 1994.

[4] S L Woronowicz. Differential calculus on compact matrix pseudogroups (quantum groups). *Commun. Math. Phys.*, 122(1):125–170, March 1989.

Acknowledgements

It is a part of the thesis written under the supervision of Professor Andrzej Borowiec.