A Hamiltonian Formalism for Perturbed (Interiors of) Nonrotating Black Holes

JERZY LEWANDOWSKI MEMORIAL CONFERENCE, 15 September 2025

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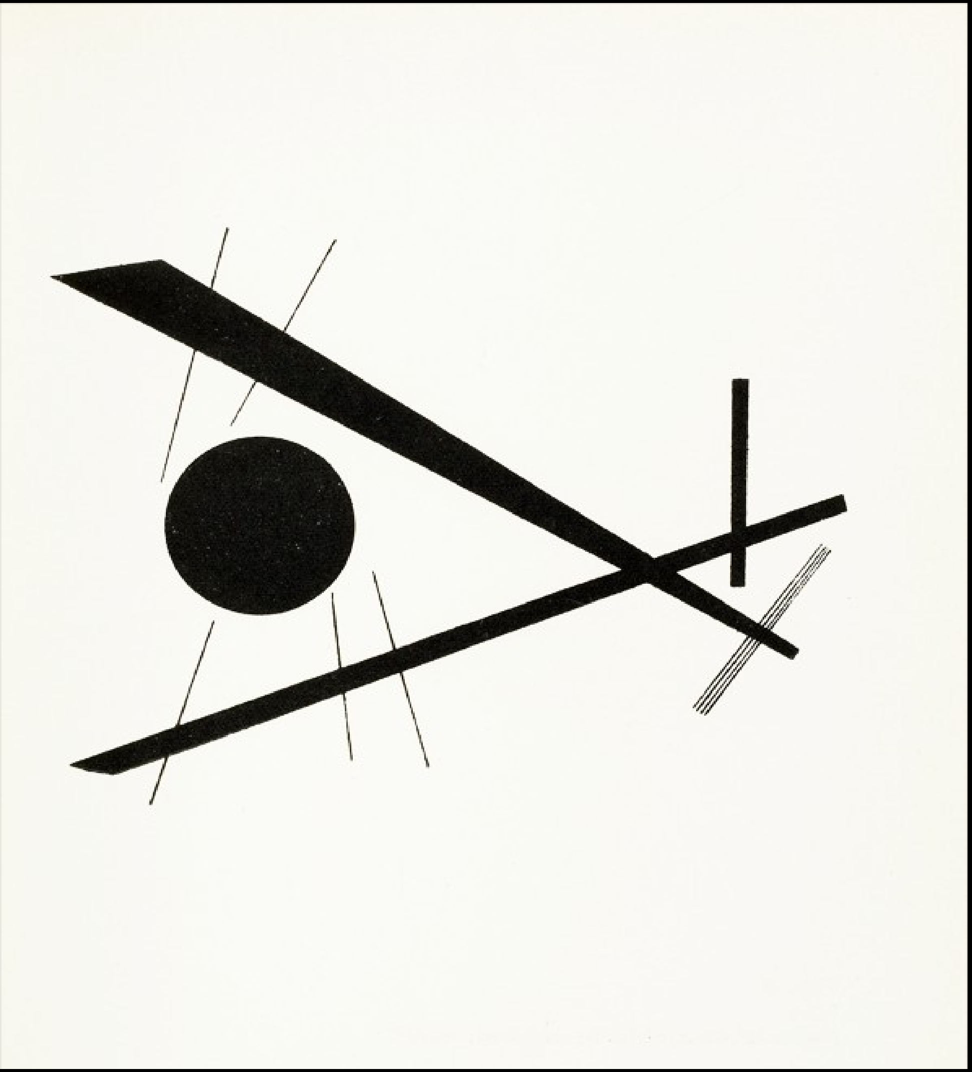


Non-Expanding Horizons in the Theory of Black Holes and Gravitational Waves
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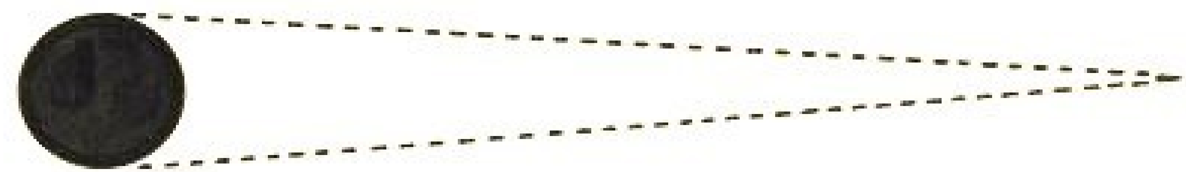
Introduction



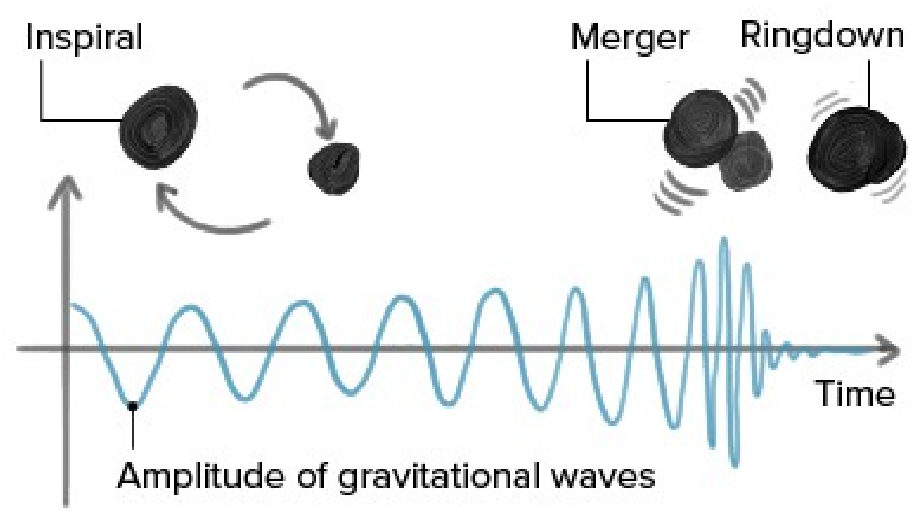
- Black hole spacetimes are a challenge for classical, semiclassical, and quantum gravity.

- **<u>PERTURBATIONS</u>** of black holes are crucial to analyze their stability.

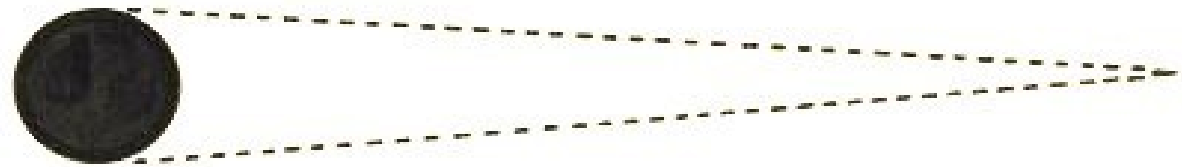
- They also have applications in astrophysics. For instance, they describe some regimes in the evolution of a black hole merger.

- This connects with the emission of gravitational waves.

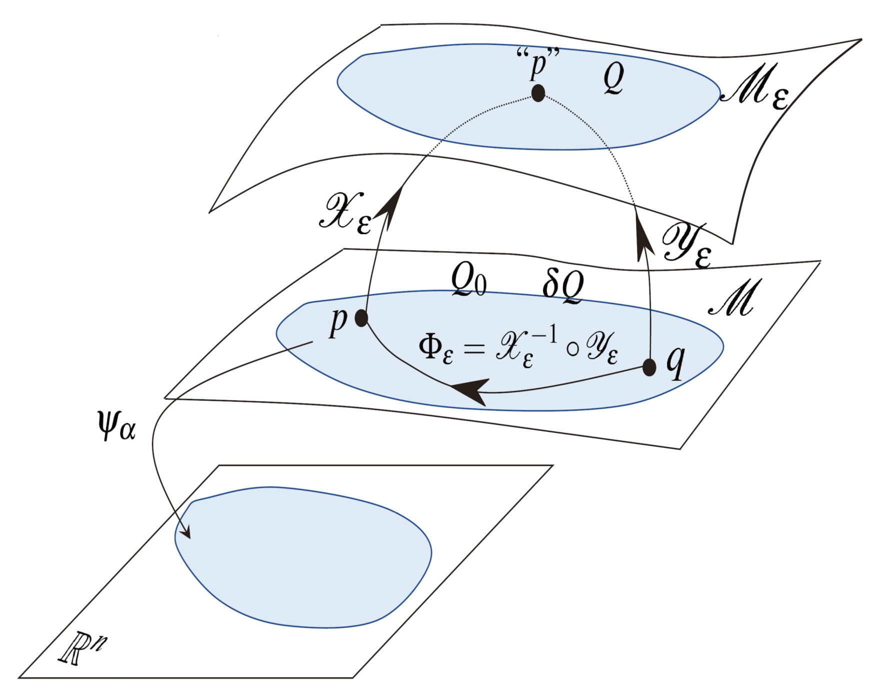
- The ringdown of perturbed black holes is dominated by quasinormal modes.



Introduction



- Identification of points of the original and the perturbed manifold introduce some gauge freedom.
- Only perturbative quantities invariant under this freedom are physical.
- These are the **PERTURBATIVE GAUGE INVARIANTS**.
- At first order, they are linear in the perturbations and can be multiplied by any background-dependent factor.



- They satisfy second-order differential equations, defined in the set of orbits of spherical symmetry. Quasinormal modes solve these equations with outgoing boundary conditions.

Introduction

- There exists an intriguing relation between different perturbative gauge invariants, given by **DARBOUX TRANSFORMATIONS**.

- Suppose that φ satisfies a wave equation in two dimensions for a potential v_l, which depends on the angular-momentum number l.

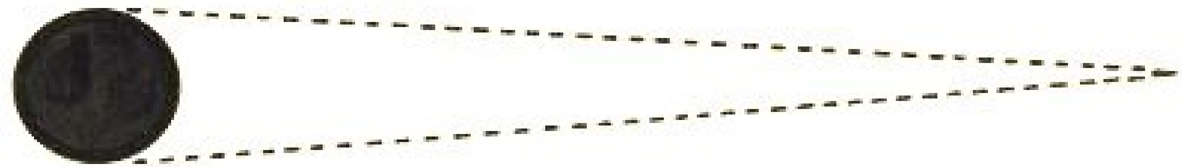
  Consider the transformation to $\Psi = \acute{\varphi} + g_l \varphi$, where the acute stands for the derivative wrt. a tortoise “radial” coordinate, and g_l satisfies the Riccati equation $\acute{g}_l + g_l^2 + v_l = c_l$, with c_l a constant.

  Then Ψ is a solution to the equation for the new potential $V_l = v_l + 2\acute{g}_l$.

- Given a solution $\acute{\acute{\varphi}}_0 + v_l \varphi_0 = -\omega_0^2 \varphi_0$, define $g_l = (\ln(\varphi_0))'$, with $c_l = \omega_0^2$.

  Then, the old and new potentials admit **isoespectral** solutions (with the same “frequency”), related by $\Psi = [\acute{\varphi}\varphi_0 - \varphi\acute{\varphi}_0]/\varphi_0$.

Introduction

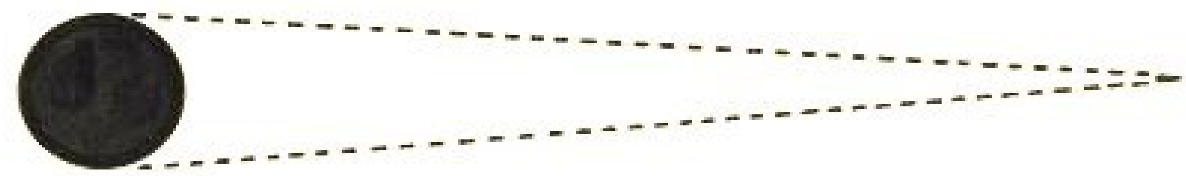


- Most of the studies have been carried out in the Lagrangian formalism.
- A **HAMILTONIAN** formulation for perturbed nonrotating black holes -as well as a higher-order perturbative formalism- was developed by Martín-García, Brizuela and G.A.M.M. in the 2000s.
- This formulation employs spherical symmetry as a key ingredient. It splits the 4-dimensional manifold into two 2-dimensional ones.
- Perturbative gauge invariants are easily characterized because they commute with the generators of perturbative diffeomorphisms.
- The Hamiltonian formulation is especially suitable for quantization.
- However, the radial dependence highly complicates the analysis.

Introduction

- The complications with the radial dependence can be handled in the **interior** of the black hole, where it becomes a time dependence.
- This interior is isometric to a Kantowski-Sachs (KS) cosmology.
- Can the Hamiltonian formulation be completed in this interior? **YES.** (*Mínguez-Sánchez & G.A.M.M.*).
- And quantum mechanically? **YES.** In LQC, the singularity is solved. (*Elizaga Navascués, Mínguez-Sánchez & G.A.M.M.*).
- Can we use it to understand Darboux transformations?

Background



- The metric in the interior can be written in terms of **triad** variables as

$$ds^2 = p_b^2(\tau)\left(-\underline{N}^2(\tau)\ |p_c(\tau)|\ d\tau^2 + \frac{dx^2}{|p_c(\tau)|}\right) + |p_c(\tau)|\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

with extrinsic-curvature variables such that $\{b, p_b\} = 1,\ \{c, p_c\} = 2.$

- The transformation $(p_b, b) \rightarrow i(\bar{p}_b, -\bar{b})$ interchanges the time role, $\tau \leftrightarrow x.$

- The KS background is subject ONLY to the **Hamiltonian** constraint

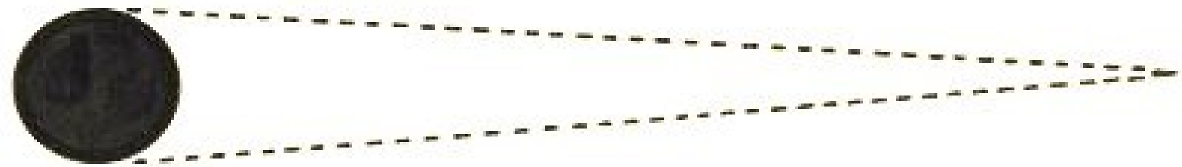
$$\underline{N}\, H_{KS} = -\frac{\underline{N}}{2}\left(\Omega_b^2 + 2\,\Omega_b \Omega_c + p_b^2\right), \quad \Omega_j = j\, p_j, \quad j = b, c.$$

The Omega-variables are generators of dilations.

- For classical solutions of mass M in "Schwarzschild" time $\tau \rightarrow T$:

$$p_b^2 = -\underline{N}^{-1} = T(2M - T), \quad |p_c| = T^2, \quad \Omega_b = T - 2M, \quad \Omega_c = M.$$

Perturbations



- We consider compact sections with the topology of $S^1 \times S^2$. Then, zero-modes are isolated and can be treated exactly.

- We expand our perturbations in REAL spherical harmonics and **Fourier modes**.

- We use a real Regge-Wheeler-Zerilli basis of harmonics.

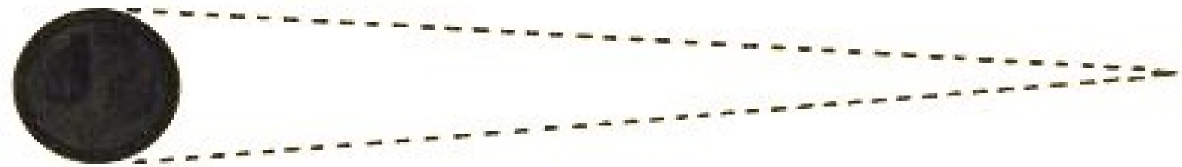
- Spherical harmonics split in polar and axial under parity.

- A polar harmonic of eigenvalue $-l(l+1)$ for the Laplacian on S^2 has parity eigenvalue equal to $(-1)^l$. Scalar harmonics Y_l^m are polar.

- Using capital Latin letters for S^2-indices, we decompose any symmetric tensor as

$$T_{ab}\,dx^a\,dx^b = T_{xx}\,dx^2 + 2\mathrm{T}_{xA}\,dx\,dx^A + T_{AB}\,dx^A\,dx^B.$$

Perturbations



- For scalars on S^2, we have $\zeta(\theta,\phi)=\sum \zeta_l^m Y_l^m .$
- For covectors, $w_A(\theta,\phi)=\sum\left(W_l^m Z_{l\ A}^m + w_l^m X_{l\ A}^m\right),$

  where we include polar and axial contributions.

  Using the metric γ_{AB} on S^2 and its covariant derivative, we have

$$Z_{l\ A}^m = Y_{l\ :A}^m , \quad X_{l\ A}^m = \epsilon_{AB}\gamma^{BC} Y_{l\ :C}^m , \quad l \geq 1 .$$

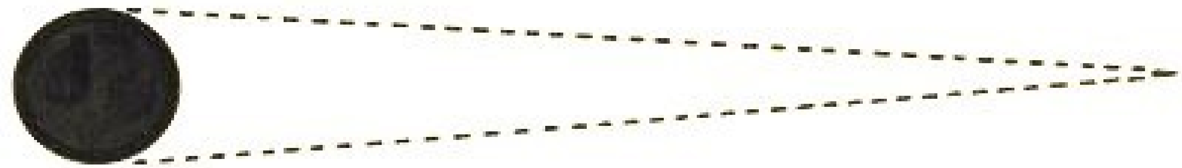
- Finally, for tensors

$$T_{AB}(\theta,\phi)=\sum \tilde{T}_l^m \gamma_{AB} Y_l^m + \sum\left(T_l^m Z_{l\ AB}^m + t_l^m X_{l\ AB}^m\right),$$

with $X_{l\ AB}^m = \frac{1}{2}\left(X_{l\ A:B}^m + X_{l\ B:A}^m\right), \quad Z_{l\ AB}^m = Y_{l\ :AB}^m + \frac{l(l+1)}{2}\gamma_{AB} Y_l^m , \quad l \geq 2 .$

All these harmonics are “orthonormalized”.

Perturbations



- We choose real spherical harmonics,

$$Y_l^m \rightarrow \left\{ Y_l^m,\ m=0;\ \ \frac{(-1)^m}{\sqrt{2}}\left(Y_l^m + Y_l^{m*}\right),\ m>0;\ \ \frac{(-1)^m}{i\sqrt{2}}\left(Y_l^{|m|} - Y_l^{|m|*}\right),\ m<0 \right\}.$$

- Similarly, for the Fourier expansion on S^1, we employ real modes,

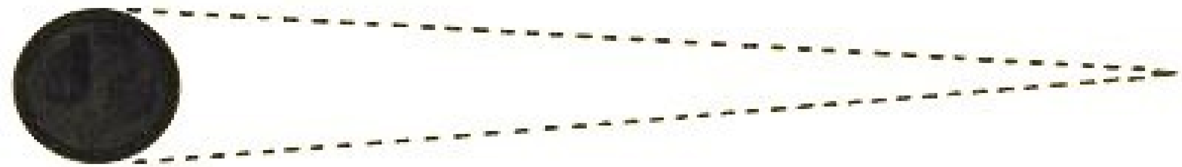
$$W_{n,\lambda} \rightarrow \left\{ W_0 = 1;\ \ W_{n,+} = \sqrt{2}\cos\omega_n x,\ \ W_{n,-} = \sqrt{2}\sin\omega_n x,\ \ \omega_n = 2\pi n,\ n \geq 1 \right\}.$$

- For simplicity, we will restrict ourselves to AXIAL perturbations with $l \geq 2$. Polar perturbations can be studied along similar lines.

- There are no scalar axial perturbations, and vector ones are pure gauge.

- We might include a perturbative **scalar field** in the analysis. But it would only contribute with polar perturbations.

Perturbations



- Calling $\{\nu\}=\{n,\lambda=\pm,l,m\}$, we can expand the axial pertubations of the spatial metric, its momentum, and the shift vector as

$$\Delta h_{ab}dx^a dx^b = -2\sum h_1^\nu(\tau) X^m_{l\ A}(\theta,\phi) W_{n,\lambda}(x)\, dx\, dx^A + \sum h_2^\nu(\tau) X^m_{l\ AB}(\theta,\phi) W_{n,\lambda}(x)\, dx^A dx^B,$$

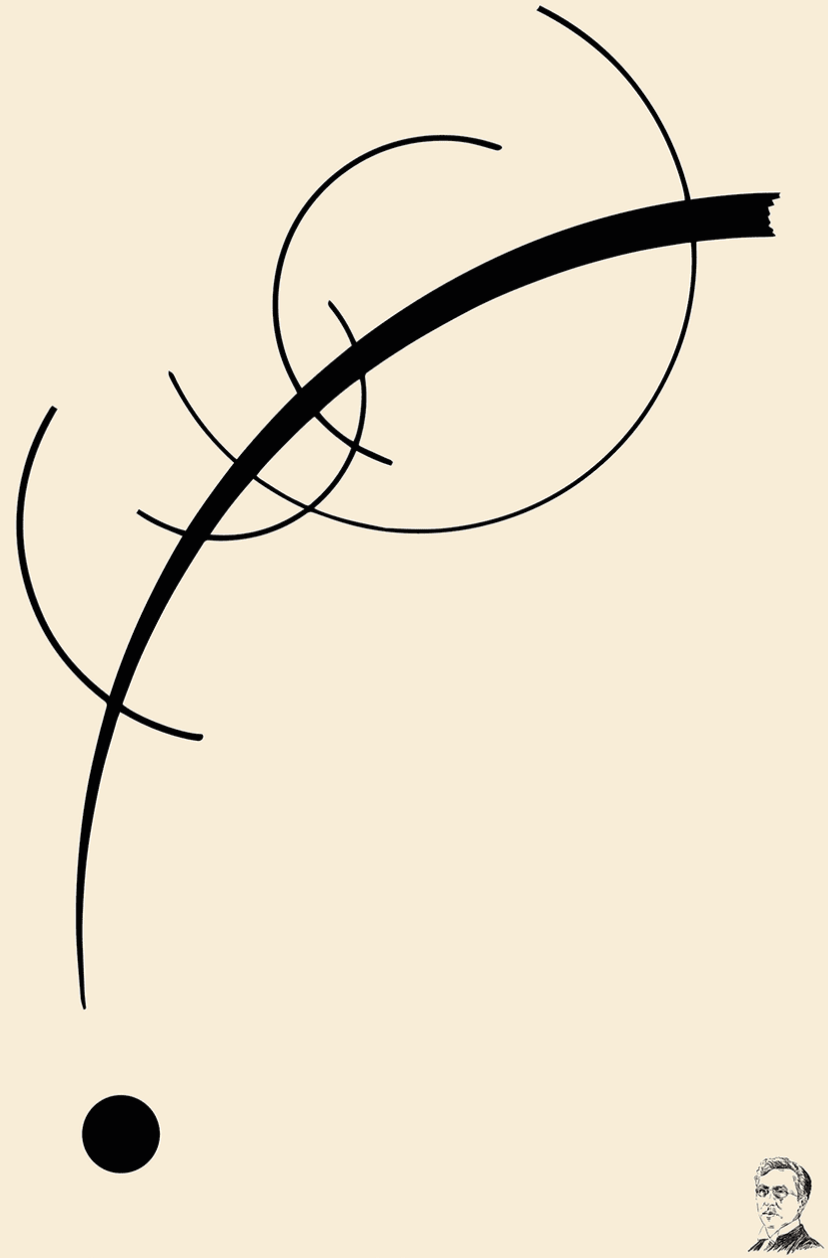
$$\Delta\left[\frac{p_{ab}}{\sqrt{h}} dx^a dx^b\right] = -\frac{4\pi p_b^2}{V}\sum \frac{p_1^\nu(\tau)}{l(l+1)} X^m_{l\ A}(\theta,\phi) W_{n,\lambda}(x)\, dx\, dx^A$$
$$+\frac{8\pi p_c^2}{V}\sum \frac{p_2^\nu(\tau)}{l(l+1)(l+2)} X^m_{l\ AB}(\theta,\phi) W_{n,\lambda}(x)\, dx^A dx^B,$$

$$N_a dx^a = -16\pi \sum h_0^\nu(\tau) X^m_{l\ A}(\theta,\phi) W_{n,\lambda}(x)\, dx^A.$$

- At second order, the contribution of the perturbations to the action has the form

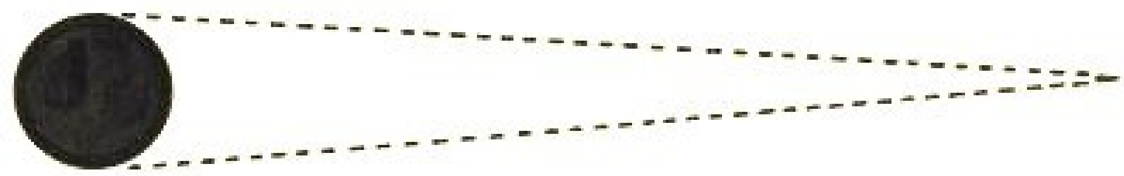
$$\frac{1}{16\pi}\int d\tau \sum\left(\dot{h}_1^\nu p_1^\nu + \dot{h}_2^\nu p_2^\nu - h_0^\nu C_\nu^{ax} - \underline{N}\,\underline{H}_\nu^{ax}\right).$$

Perturbative diff. constraints   Hamiltonian constraint



Free Curve to the Point • Accompanying Sound of Geometric Curves • 1925 • Wassily Kandinsky

Gauge invariants



- Considering the background as fixed, we can perform a linear canonical transformation in the perturbations so that they are described by gauge invariant canonical pairs, and by the perturbative constraints and variables conjugate to them,

$$\{h_1^\nu, p_1^\nu, h_2^\nu, p_2^\nu\} \to \left\{\tilde{Q}_1^\nu, \tilde{P}_1^\nu, \tilde{Q}_2^\nu, \tilde{P}_2^\nu = -\frac{1}{2} C_\nu^{ax}\right\}.$$

  E.g. with the generating function

$$F^\nu = h_1^\nu \tilde{Q}_1^\nu + h_2^\nu \tilde{P}_2^\nu - \frac{\lambda \omega_n}{2} h_2^\nu \tilde{Q}_1^\nu + \frac{(l+2)!}{4(l-2)!} \frac{(\Omega_b + \Omega_c)}{p_c^2} (h_2^\nu)^2 - 2l(l+1) \frac{\Omega_b}{p_b^2} \left(\frac{\omega_n^2}{4} (h_2^\nu)^2 + \lambda \omega_n h_1^\nu h_2^\nu \right).$$

- The perturbative term in the Hamiltonian changes due to the background evolution of the generating function, given by its Poisson bracket with the background Hamiltonian.

Gauge invariants

- The perturbative contribution to the action can be written

$$\int d\tau \left\{ \left(\underline{N} - \underline{\tilde{N}} \right) H_{KS} + \frac{1}{16\pi} \sum \left(\dot{\tilde{Q}}_1^{\nu} \tilde{P}_1^{\nu} + \dot{\tilde{Q}}_2^{\nu} \tilde{P}_2^{\nu} \right) + \frac{1}{8\pi} \sum \tilde{h}_0^{\nu} \tilde{P}_2^{\nu} - \underline{\tilde{N}} \sum \underline{\tilde{H}}_{\nu}^{ax} \right\},$$

  where the new lapse includes quadratic perturbative terms and

$$\underline{\tilde{H}}_{\nu}^{ax} = \frac{p_b^2 \left(\tilde{Q}_1^{\nu} \right)^2}{2l(l+1)} + \frac{l(l+1)}{2 p_b^2} \left[8\Omega_b^2 + 8\Omega_b \Omega_c + 4 p_b^2 + (l+2)(l-1) p_b^2 \right] \left(\tilde{P}_1^{\nu} \right)^2$$

$$+ \frac{(l-2)!}{2(l+2)!} \omega_n^2 p_c^2 \left[\tilde{Q}_1^{\nu} + \frac{4l(l+1)}{p_b^2} \Omega_b \tilde{P}_1^{\nu} \right]^2 + 2\Omega_b \tilde{Q}_1^{\nu} \tilde{P}_1^{\nu}.$$

- We can eliminate the cross-terms in the perturbative contribution to the Hamiltonian and introduce (up to a constant factor) the Gerlach-Sengupta gauge invariant, **generalized to any background** and **evaluated in the interior**.

Gerlach-Sengupta

- This gauge invariant and its momentum are given by

$$Q_{GS}^{\nu} = -\sqrt{\frac{(l-2)!}{(l+2)!}}\left[\tilde{Q}_1^{\nu} + \frac{4l(l+1)}{p_b^2}\Omega_b \tilde{P}_1^{\nu}\right],$$

$$P_{GS}^{\nu} = -\sqrt{\frac{(l+2)!}{(l-2)!}}\tilde{P}_1^{\nu} + 2\Omega_b\sqrt{\frac{(l-2)!}{(l+2)!}}\left[\tilde{Q}_1^{\nu} + 41(l+1)\frac{1}{p_b^2}\Omega_b \tilde{P}_1^{\nu}\right].$$

- After this canonical transformation, the perturbative term of the Hamiltonian adopts a simple form (easy to quantize),

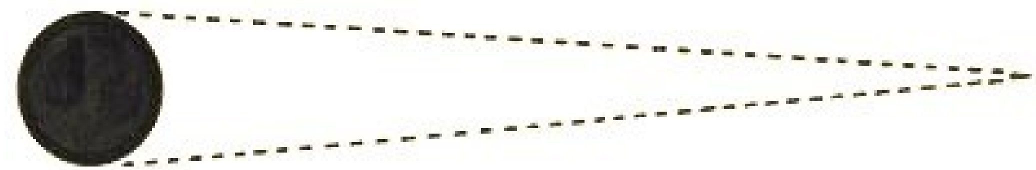
$$H_{\nu}^{ax,(GS)} = \frac{1}{2}(P_{GS}^{\nu})^2 + \frac{\tilde{V}}{2}(Q_{GS}^{\nu})^2,$$

$$\tilde{V} = \omega_n^2 p_c^2 + l(l+1)p_b^2 - 4(\Omega_b^2 + p_b^2) \simeq \omega_n^2 p_c^2 - l(l+1)(\Omega_b^2 + 2\Omega_b\Omega_c) + 8\Omega_b\Omega_c.$$

- In **2-dimensions**, the generalized Gerlach-Sengupta master variable for **any background** is

$$Q_{GS}^{lm}(\tau, x) = \sum_{n,\lambda} Q_{GS}^{\nu}(\tau) W_{n,\lambda}(x).$$

Master equation



- The Gerlach-Sengupta modes satisfy

$$\left[\left(\left(\underline{N}^{-1}\partial_\tau\right)^2+\omega_n^2\, p_c^2\right)+\left(l\,(l+1)\,p_b^2-4\,\Omega_b^2-4\,p_b^2\right)\right]Q^{\nu}_{GS}=0.$$

  Using the Laplacian of the 2-dimensional metric induced on the set of spherical orbits and the Gerlach-Sengupta master variable, we obtain

$$\left[\Box_2+\frac{|p_c|}{p_b^2}\left(l\,(l+1)\,p_b^2-4\,\Omega_b^2-4\,p_b^2\right)\right]Q^{lm}_{GS}(\tau,x)=0.$$

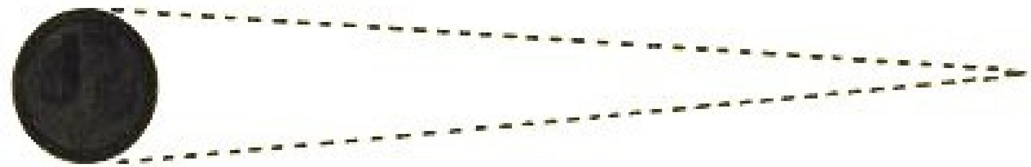
- It is easy to **reconstruct** the metric perturbations with this invariant.

- The above contribution of ω_n^2 is not constant, but appears multiplied by p_c^2.

- We can render it constant with the **Cunningham-Price-Moncrief** invariant,

$$\mathbb{Q}^{\nu}_{CPM}=2\sqrt{\frac{(l-2)!}{(l+2)!}}\,Q^{\nu}_{CPM}.$$

Master equation



- We carry out the **canonical** transformation

$$Q^{\nu}_{CPM} = \sqrt{|p_c|}\, Q^{\nu}_{GS}, \qquad P^{\nu}_{CPM} = \frac{1}{\sqrt{|p_c|}}\left(P^{\nu}_{GS} + \frac{1}{2}\frac{\{|p_c|, H_{KS}\}}{|p_c|} Q^{\nu}_{GS}\right).$$

Background Hamiltonian

- We could use the **Regge-Wheeler** master variable instead, $2\,\mathbb{Q}^{lm}_{RW} = \partial_x \mathbb{Q}^{lm}_{CPM}$.

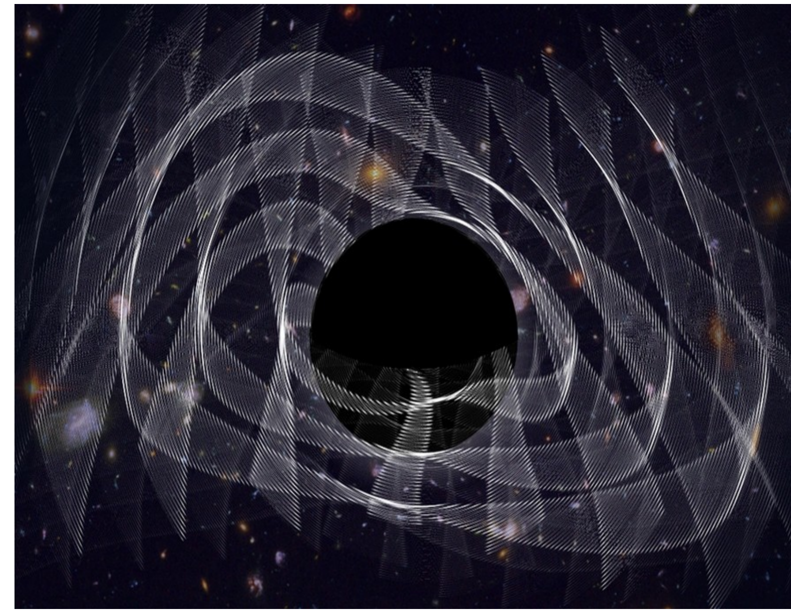
- The new perturbative contribution to the Hamiltonian constraint is

$$H^{ax}_{\nu}[N] = \frac{N|p_c|}{2}\left[(P^{\nu}_{CPM})^2 + \left(\omega_n^2 - \frac{1}{p_c^2}\left[3\left(\Omega_b^2 + p_b^2\right) - l(l+1)\,p_b^2\right]\right)(Q^{\nu}_{CPM})^2\right].$$

- This leads to the master equation $\left[\Box_2 - \left(\frac{l(l+1)}{|p_c|} - 3\frac{\Omega_b^2 + p_b^2}{p_b^2\,|p_c|}\right)\right] Q^{lm}_{CPM} = 0.$

For Schwarzschild: $\left[\Box_2 - \left(\frac{l(l+1)}{T^2} - \frac{6M}{T^3}\right)\right] Q^{lm}_{CPM} = 0.$

Darboux



- Defining $d\bar{\tau} = \underline{N}\left|p_c\right| d\tau ,$ the equation can be written

$$\left(\partial_{\bar{\tau}}^2 + \omega_n^2 - \frac{1}{p_c^2}\left[3\left(\Omega_b^2 + p_b^2\right) - l\,(l+1)\,p_b^2\right]\right) Q^{\nu}_{CPM} = 0.$$

- For the time $\bar{\tau},$ the perturbative Hamiltonian has the form

$$\bar{H}^{ax}_{\nu} = \frac{1}{2}\left(P^{\nu}_{CPM}\right)^2 + \frac{1}{2}\left(\omega_n^2 + v_l\right)\left(Q^{\nu}_{CPM}\right)^2 ,$$

and the gauge invariant satisfies $Q''^{\nu}_{CPM} + \left(\omega_n^2 + v_l\right) Q^{\nu}_{CPM} = 0 ,$ where $f' = \partial_{\bar{\tau}} f .$

- RECALL:

A **Darboux** transformation $Q^{\nu}_{CPM} = \bar{Q}'^{\nu} + g_l \bar{Q}^{\nu}$ leads to the new equation $\bar{Q}''^{\nu} + \left(\omega_n^2 + V_l\right)\bar{Q}^{\nu} = 0 ,$ with $V_l = v_l + 2 g_l' ,$ if $g_l' + g_l^2 + v_l = c_l .$

Canonical Darboux

- We want to show that Darboux transformations are just **canonical transformations** that respect the structure of the Hamiltonian!

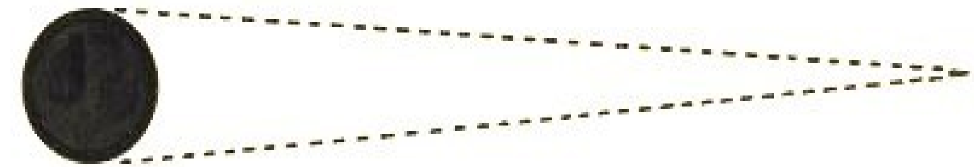
- Consider a generic canonical transformation,

$$\left.\begin{array}{l} Q^{\nu}_{CPM} = A\bar{Q}^{\nu} + B\bar{P}^{\nu}, \qquad P^{\nu}_{CPM} = C\bar{Q}^{\nu} + D\bar{P}^{\nu}, \\ \text{with } AD - BC = 1. \qquad \text{(Canonical!)} \end{array}\right\}$$

- The coefficients of the transformation are background (and thus time-) dependent. We assume $B \neq 0,$ so that the transformation is not just a simple redefinition of the gauge invariant. With a (background-dependent) scaling, we set B to be constant.

- Owing to the transformation, we get a new perturbative contribution to the Hamiltonian. We require cross-terms to vanish and the coefficient of the squared momentum be one half, **as before the transformation**!

Canonical Darboux



- Canonical transformation:

$$Q^{\nu}_{CPM} = A\,\bar{Q}^{\nu} + B\,\bar{P}^{\nu}, \qquad P^{\nu}_{CPM} = C\,\bar{Q}^{\nu} + D\,\bar{P}^{\nu},$$

with $C = \dfrac{AD-1}{B}$. (Canonical!)

- Hamiltonian: $\bar{H}^{ax}_{\nu} = \dfrac{1}{2}(\bar{P}^{\nu})^2 + \dfrac{1}{2}(\omega_n^2 + V_l)(\bar{Q}^{\nu})^2.$

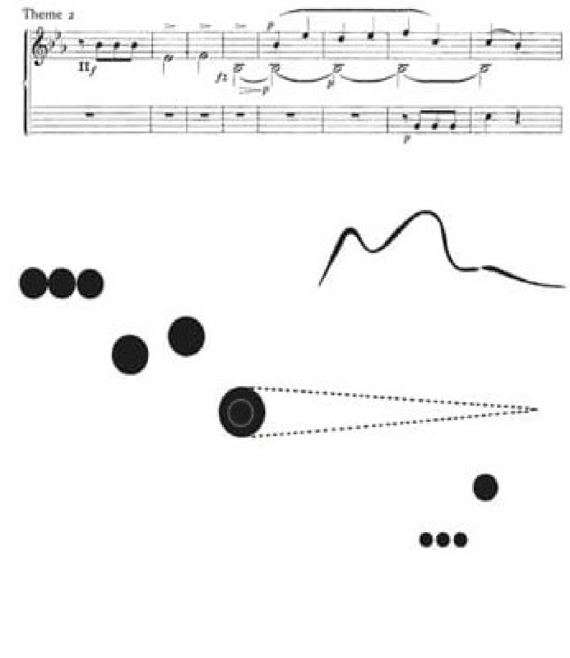
It is not difficult to see that this requires

$A = D.$ (No cross-term!)

$$\acute{g}_l + g_l^2 + (\omega_n^2 + v_l) = \frac{1}{B^2}, \text{ with } g_l = \frac{D}{B}.$$ (Momentum coeff.)

- If the new potential must not depend on ω_n, we must have $B^{-2} = \omega_n^2 + c_l$.

The transformation is **fixed** given c_l and a solution to the Riccati equation.



Theme 2 translated into points.

Darboux/Canonical

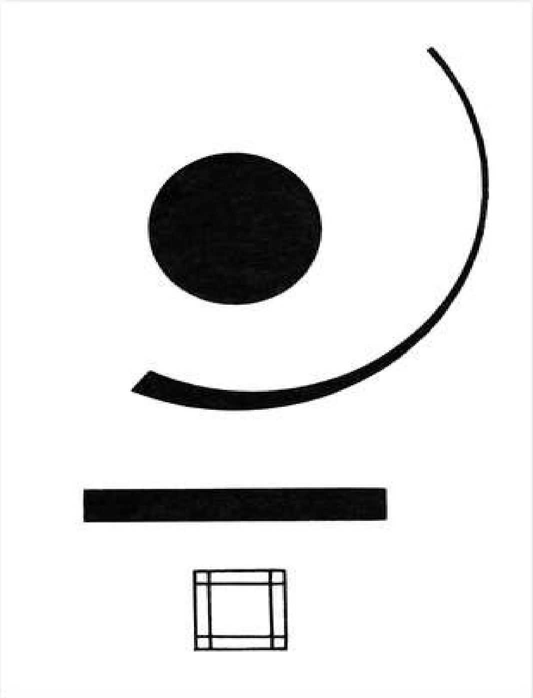
- DARBOUX: Find the gauge invariant combinations of the metric perturbations.
- Master functions are combinations of gauge invariants and first derivatives which satisfy wave equations.
- Darboux transformations relate such master functions.

- CANONICAL: First transformation to variables that commute with the perturbative constraints.
- Second transformation to variables with "generalized" harmonic oscillator Hamiltonian.
- Find the transformations preserving this Hamiltonian form.

- They are characterized by a constant c_l and a solution to $g'_l + g_l^2 + v_l = c_l$.
- Darboux corresponds (in the interior) to the **canonical transformations**

$$Q^{\nu}_{CPM} = \frac{g_l}{\sqrt{\omega_n^2 + c_l}} \bar{Q}^{\nu} + \frac{1}{\sqrt{\omega_n^2 + c_l}} \bar{P}^{\nu}, \qquad P^{\nu}_{CPM} = \left(\frac{g_l^2}{\sqrt{\omega_n^2 + c_l}} - \sqrt{\omega_n^2 + c_l} \right) \bar{Q}^{\nu} + \frac{g_l}{\sqrt{\omega_n^2 + c_l}} \bar{P}^{\nu}.$$

Conclusions



- We have developed a **Hamiltonian formalism** for perturbations of nonrotating black holes, adapted to their interior.
- We have identified gauge invariants/master variables. Their dynamics involve quasinormal modes, relevant during **ringdown**.
- Although derived from General Relativity, all relations are expressed in terms of the background, and can be extended to **effective** ones.
- The formalism allows for an almost direct **quantization**, for example in (hybrid) LQC!
- Darboux transformations become **canonical transformations**!
- We can now investigate whether isospectroscopy is realized as a true unitary transformation in quantum field theory (including all l 's).

In Memoriam