Scalar curvature operator: Applications to models of LQG dynamics

Ilkka Mäkinen

National Centre for Nuclear Research, Poland

Jerzy Lewandowski Memorial Conference

Warsaw, 19.09.2025

The kinematical framework of loop quantum gravity

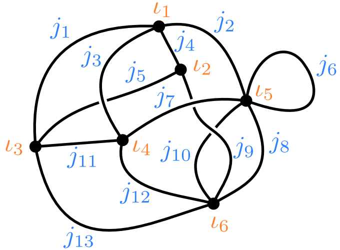
Canonical quantization of general relativity in the Ashtekar formulation. The elementary variables in the classical theory are

$$h_e = \mathcal{P}\exp\left(-\int_e A\right) \qquad E(S) = \int_S d^2\sigma\, n_a(\sigma)E^a_i(\sigma)\tau^i$$

i.e. holonomies of the Ashtekar connection along 1D curves and fluxes of the densitized triad through 2D surfaces.

The kinematical Hilbert space of LQG is spanned by the spin network states

$$\big|\Gamma, \{j_e\}, \{\iota_v\}\big\rangle \;=\; \left(\prod_{v\in\Gamma}\iota_v\right)\cdot\left(\prod_{e\in\Gamma}D^{(j)}(h_e)\right)$$



The holonomy operator acts as a multiplicative operator in the spin network representation. The flux operator acts essentially as a functional derivative with respect to the connection.

Dynamics in canonical loop quantum gravity

In the canonical formulation of LQG, the dynamics is governed by the Hamiltonian constraint operator

$$\hat{C}(N)$$

Within the canonical theory there are two main approaches to the dynamics; accordingly the above operator can be interpreted in two different ways:

- As a Hamiltonian constraint for the vacuum theory, where it determines the space of physical states through the condition

$$\hat{C}(N)|\Psi\rangle = 0$$

- As a physical Hamiltonian for gravity coupled to a reference matter field.

Classical theory: $\frac{d}{dT}F(A,E) = \{F, H_{\rm phys}\}$

Quantum theory: $i\frac{d}{dT}|\Psi(T)\rangle = \hat{H}_{\rm phys}|\Psi(T)\rangle$

For a suitable matter field (irrotational dust): $\hat{H}_{\rm phys} = \hat{C}(1)$

[Brown, Kuchař 1994; Giesel, Thiemann 2012; Husain, Pawłowski 2013]

Scalar curvature in loop quantum gravity

The object of interest: Ricci scalar integrated over the spatial manifold

$$\int_\Sigma d^3x\,\sqrt{q}\,{}^{(3)}R$$

Relevant to loop quantum gravity both as a geometrical observable characterizing the geometry of the spatial manifold, and as a possible Lorentzian part of the Hamiltonian constraint:

$$C = \frac{1}{\beta^2}\frac{\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab}}{\sqrt{|\det E|}} + \frac{1+\beta^2}{\beta^2}\sqrt{|\det E|}\,{}^{(3)}R$$

In this talk: Direct quantization of the Ricci scalar as a function of the Ashtekar variables (restricted to a fixed cubic graph) [Lewandowski, I.M. 2022]

Other Ricci scalar operators available in the literature:

- Using Regge's formula for the integral $\int d^3x\,\sqrt{q}\,{}^{(3)}R$ in terms of hinge lengths and deficit angles [Alesci, Assanioussi, Lewandowski 2014]
- Using an expression of the Ricci scalar in terms of the spin connection and "twisted geometry" variables [Long, Liu 2024]

Ricci scalar as a function of the Ashtekar variables

The starting point of our construction is to express the Ricci scalar directly as a function of the Ashtekar variables.

$$ {}^{(3)}R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \qquad q^{ab} = \frac{E^a_i E^b_i}{|\det E|} $$

A straightforward (but rather long) calculation gives an explicit expression of the form

$$ \sqrt{|\det E|}\,{}^{(3)}R = f\left(E^a_i, \partial_a E^b_i, \partial_a \partial_b E^c_i\right) $$

The partial derivatives of the triad are problematic if one wishes to obtain a gauge invariant operator (under internal SU(2) gauge transformations) upon quantization. It is better to use the gauge covariant derivatives

$$ \mathcal{D}_a E^b_i = \partial_a E^b_i + \epsilon_{ij}{}^k A^j_a E^b_k $$

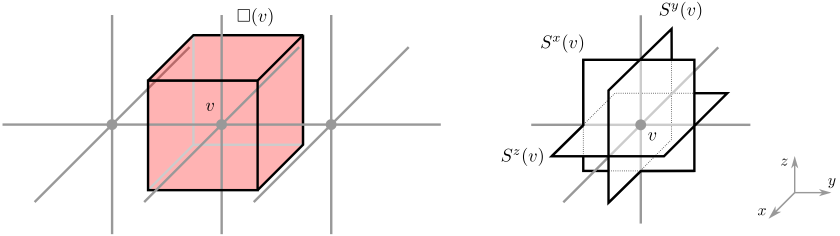
It turns out that the partial derivatives can be substituted with covariant derivatives "for free":

$$ \sqrt{|\det E|}\,{}^{(3)}R = f\left(E^a_i, \mathcal{D}_a E^b_i, \mathcal{D}_{(a}\mathcal{D}_{b)} E^c_i\right) $$

Regularization on a cubic graph

The integrated Ricci scalar is regularized as a Riemann sum over a cubic partition:

$$\int d^3x \sqrt{q}\, {}^{(3)}R \simeq \sum_{\square} \epsilon^3 \sqrt{|\det E|}(v)\, {}^{(3)}R(v)$$



The covariant derivatives of the triad can be discretized as finite differences of parallel transported flux variables (also known as gauge covariant fluxes):

$$\widetilde{E}(S, x_0) = \int_S d^2\sigma\, n_a(\sigma) h_{x(\sigma)\to x_0} E^a\big(x(\sigma)\big) h^{-1}_{x(\sigma)\to x_0}$$

The holonomies $h_{x(\sigma)\to x_0}$ perform parallel transport from points on the surface to a fixed point x_0 along a chosen family of paths.

Quantization

After regularization on the cubic lattice, the Ricci scalar has been expressed as

$$\int d^3x\,\sqrt{q}\,{}^{(3)}\!R = \sum_{\square} f\Big(E_i\big(S^a_\square\big), \Delta_a E_i\big(S^b_\square\big), \Delta_{ab} E_i\big(S^c_\square\big)\Big) + \mathcal{O}(\epsilon)$$

where $\Delta_a E_i(S^b)$ and $\Delta_{ab} E_i(S^c)$ are finite differences of parallel transported flux variables approximating the covariant derivatives of the triad.

Every factor appearing here can now be promoted into an operator in LQG. Negative powers of the volume are quantized using the following prescription (Tikhonov regularization):

$$\frac{1}{V_\square} \quad \longrightarrow \quad \widehat{\mathcal{V}_v^{-1}} := \lim_{\epsilon\to 0}\ \frac{\hat{V}_v}{\hat{V}_v^2 + \epsilon^2}$$

The result is an operator of the form

$$\int \widehat{d^3x\,\sqrt{q}\,{}^{(3)}\!R} = \sum_{v\in\Gamma_0} \hat{\mathcal{R}}_v$$

on the Hilbert space of a fixed cubic graph Γ_0.

Quantum-reduced loop gravity

The Hilbert space of quantum-reduced loop gravity is spanned by the ''reduced spin network states'' [Alesci, Cianfrani 2013]

$$\prod_{e\in\Gamma_0} D^{(j_e)}_{j_e j_e}(h_e)_{i_e}$$

They are characterized by the following properties:

- The state is defined on a cubic graph Γ_0
- The magnetic indices of each holonomy are maximal ($m_e = j_e$) with respect to the basis diagonalizing $\hat{J}^2$ and $\hat{J}_{i_e}$, where $i_e = x, y, z$ is chosen according to the direction of the edge e
- In the original formulation of the model the spins are assumed to be large: $j_e \gg 1$ for every edge e

Quantum-reduced loop gravity = ''LQG in diagonal gauge'' ($E^a_i = 0$ for $a \neq i$)
[Alesci, Cianfrani, Rovelli 2013; I.M. 2023]

$$\hat{M} \;:\; \text{''}\,\widehat{E^a_{i\neq a} = 0}\,\text{''} \qquad\qquad \hat{M}|\Psi_0\rangle \;\to\; 0 \quad \text{(for large } j\text{)}$$

Operators on the reduced Hilbert space

For a large class of loop quantum gravity operators, the action of the operator on a reduced spin network state $|\Psi_0\rangle$ has the structure [I.M. 2020]

$$\hat{\mathcal{O}}|\Psi_0\rangle \ = \ f(j)|\Psi\rangle + g(j)|\Phi\rangle$$

where $|\Psi\rangle \in \mathcal{H}_{\rm reduced}$, and for large j,

$$f(j) \gg g(j)$$

This suggests that operators of quantum-reduced loop gravity can be obtained from operators of full loop quantum gravity by dropping the small "offending" terms, defining the reduced operator ${}^R\hat{\mathcal{O}}$ as

$${}^R\hat{\mathcal{O}}|\Psi_0\rangle := f(j)|\Psi\rangle$$

The reduced operator ${}^R\hat{\mathcal{O}}$ is:

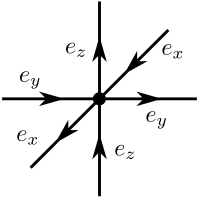
- A well-defined operator on the reduced Hilbert space
- A good approximation of the action of the full operator $\hat{\mathcal{O}}$ on the state $|\Psi_0\rangle$:

$$\frac{\big|\big|\hat{\mathcal{O}}|\Psi_0\rangle - {}^R\hat{\mathcal{O}}|\Psi_0\rangle\big|\big|}{\big|\big|\hat{\mathcal{O}}|\Psi_0\rangle\big|\big|} \ll 1$$

- Typically very simple in comparison with the corresponding full operator

The one-vertex model

Although the action of the Hamiltonian is explicitly computable on the entire reduced Hilbert space, we will now consider a very simple model, which is obtained by choosing a graph containing just a single six-valent node.



We assume that the spatial manifold has the topology of a three-torus (or has periodic boundary conditions) so the graph is formed by three closed mutually orthogonal edges.

The state space of the model is spanned by the basis states

$$|j_x j_y j_z\rangle = D^{(j_x)}_{j_x j_x}(h_{e_x})_x D^{(j_y)}_{j_y j_y}(h_{e_y})_y D^{(j_z)}_{j_z j_z}(h_{e_z})_z$$

Hamiltonian in the one-vertex model

To obtain the Hamiltonian constraint for the single-vertex model, we compute

$$\hat{C}(N)|j_x j_y j_z\rangle = {}^R\hat{C}(N)|j_x j_y j_z\rangle \;+\; \text{lower order}$$

The result is

$${}^R\hat{C}_E(N) = -\frac{1}{\beta^2}N(v)\left[\sqrt{\frac{\hat{j}_x\hat{j}_y}{\hat{j}_z}}\,\hat{s}_x^{(1)}\hat{s}_y^{(1)} + \sqrt{\frac{\hat{j}_y\hat{j}_z}{\hat{j}_x}}\,\hat{s}_y^{(1)}\hat{s}_z^{(1)} + \sqrt{\frac{\hat{j}_z\hat{j}_x}{\hat{j}_y}}\,\hat{s}_z^{(1)}\hat{s}_x^{(1)}\right]$$

$${}^R\hat{C}_L(N) = -16\frac{1+\beta^2}{\beta^2}N(v)\left[\frac{\hat{j}_x^{3/2}}{\sqrt{\hat{j}_y\hat{j}_z}}\big(\hat{s}_x^{(1/2)}\big)^4 \;+\; \text{cycl. perm.}\right]$$

where

$$\hat{j}_i|j_x j_y j_z\rangle = j_i|j_x j_y j_z\rangle$$

$$\hat{s}_x^{(k)}|j_x j_y j_z\rangle = \frac{1}{2i}\Big(|j_x+k, j_y, j_z\rangle - |j_x-k, j_y, j_z\rangle\Big), \quad \text{etc.}$$

Analogy with loop quantum cosmology

Consider the classical phase space of a homogeneous and isotropic universe:

$$A_a^i = c(t)\delta_a^i \qquad E_i^a = p(t)\delta_i^a \qquad \{c, p\} = 1$$

In loop quantum cosmology the connection c does not exist as a well-defined operator, but the holonomy $e^{i\mu c}$ does:

$$\widehat{e^{i\mu c}}|p\rangle = |p - \mu\rangle \qquad\qquad \hat{p}|p\rangle = p|p\rangle$$

Hence the connection is quantized as (''polymerization'')

$$c \;\rightarrow\; \frac{\sin \mu c}{\mu} = \frac{1}{2i}\frac{e^{i\mu c} - e^{-i\mu c}}{\mu}$$

Applying the same procedure to the (Euclidean) Hamiltonian constraint gives the expression

$$C_E = -\frac{3}{\beta^2}\sqrt{p}c^2 \;\rightarrow\; -\frac{3}{\beta^2}\sqrt{p}\frac{\sin^2 \mu c}{\mu^2}$$

The classical expression for C_E is recovered in the limit $\mu \rightarrow 0$.

Extending the analogy to the Lorentzian term

There is a certain formal similarity between the polymerized expression

$$C_E^{(\mu)} = -\frac{3}{\beta^2}\sqrt{p}\frac{\sin^2\mu c}{\mu^2}$$

and the Euclidean part of the Hamiltonian in the one-vertex model:

$${}^R\hat{C}_E = -\frac{1}{\beta^2}\left[\sqrt{\frac{\hat{j}_x\hat{j}_y}{\hat{j}_z}}\,\hat{s}_x^{(1)}\hat{s}_y^{(1)} + \text{cycl. perm.}\right]$$

If we imagine that the same relation should hold for the Lorentzian part

$${}^R\hat{C}_L = -16\frac{1+\beta^2}{\beta^2}\left[\frac{\hat{j}_x^{3/2}}{\sqrt{\hat{j}_y\hat{j}_z}}\left(\hat{s}_x^{(1/2)}\right)^4 \; + \; \text{cycl. perm.}\right]$$

we can make a conjecture – a possible Lorentzian term for the LQC Hamiltonian

$$C_L^{(\mu)} = -48\frac{1+\beta^2}{\beta^2}\sqrt{p}\frac{\sin^4(\mu c/2)}{\mu^2}$$

(cf. a similar earlier proposal [Dapor, Liegener 2017])

Effective dynamics

Effective dynamics: Dynamics on a classical phase space generated by an effective Hamiltonian function motivated by considerations from the quantum theory.

$$H_{\rm eff} = H_{\rm gr}(p,c) + \frac{\pi_\phi^2}{2p^{3/2}}$$

Classical trajectory:

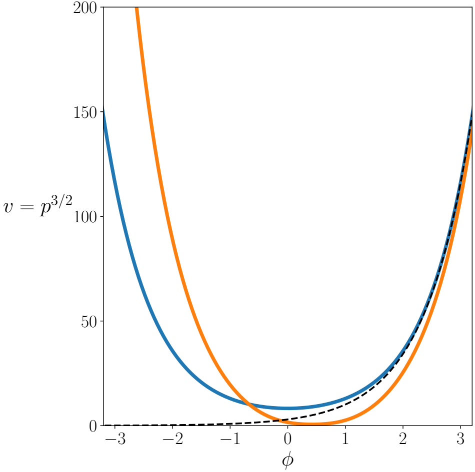
$$H_{\rm gr} = -\frac{3}{\beta^2}\sqrt{p}c^2$$

The standard Hamiltonian:

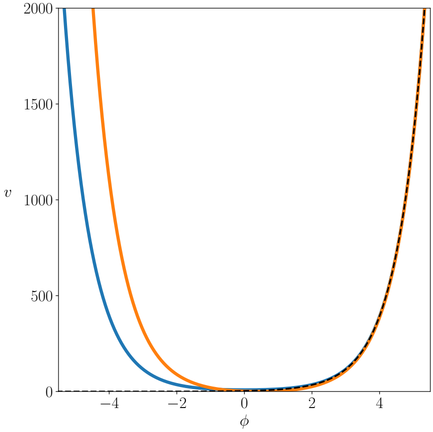
$$H_{\rm gr} = C_E^{(\mu)} = -\frac{3}{\beta^2}\sqrt{p}\frac{\sin^2 \mu c}{\mu^2}$$

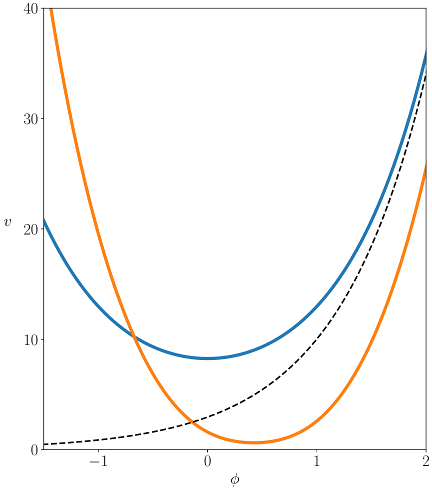
The new proposal:

$$H_{\rm gr} = C_E^{(\mu)} - 48\frac{1+\beta^2}{\beta^2}\sqrt{p}\frac{\sin^4(\mu c/2)}{\mu^2}$$



Effective dynamics





The effective dynamics of the volume agrees with the classical trajectory in the far future as well as far past of the bounce. The trajectory $v(\phi)$ is symmetric under $\phi \to \phi_{\text{bounce}} - \phi$.

$$v_{\min} = \frac{v_{\min}^{(E)}}{8(1+\beta^2)^{3/4}}$$

Quantum dynamics of semiclassical states

In the setting of the one-vertex model, we wish to study the dynamics of the states

$$|p_0, c_0\rangle_{e_x}|p_0, c_0\rangle_{e_y}|p_0, c_0\rangle_{e_z}$$

where

$$|p_0, c_0\rangle_e = \sum_j (2j+1)e^{-t(j-j_0)^2/2}e^{-ic_0 j}D^{(j)}_{jj}(h_e)_{i_e} \qquad (p_0 = j_0 + \tfrac{1}{2})$$

is a coherent state peaked on a diagonal connection and triad. (Analogous to Gaussian wave packet: $\psi_{(x_0,p_0)}(x) = \frac{1}{2\pi}\int dp\, e^{-t(p-p_0)^2/2}e^{-ipx_0}e^{ipx}$)

The evolution of the initial state $|\psi_0\rangle = |p_0, c_0\rangle_{e_x}|p_0, c_0\rangle_{e_y}|p_0, c_0\rangle_{e_z}$ is given by

$$|\psi(T)\rangle = e^{-i\hat{H}_{\rm phys}T}|\psi_0\rangle$$

We will work with irrotational dust as the physical time variable, i.e. we choose

$$\hat{H}_{\rm phys} = \hat{C}_E(1) \qquad \text{or} \qquad \hat{H}_{\rm phys} = \frac{1}{\beta^2}\hat{C}_E(1) + \frac{1+\beta^2}{\beta^2}\hat{C}_L(1)$$

The setup for numerical computations

To make the problem numerically accessible, we assume that the spins take values in the finite range $j_a \in \{j_{\min}, j_{\min}+1, \ldots, j_{\max}-1, j_{\max}\}$. (Note that the subspace of the states $|j_x j_y j_z\rangle$ with all $j_a \in \mathbb{N}$ is preserved by the Hamiltonian.)

The lower limit $j_{\min} = 1$ can be achieved by a suitable factor ordering of $\hat{H}_{\text{phys}}$, which ensures that $\langle k_x k_y k_z | \hat{H}_{\text{phys}} | j_x j_y j_z \rangle = 0$ if any $j_a = 0$ or $k_a = 0$.

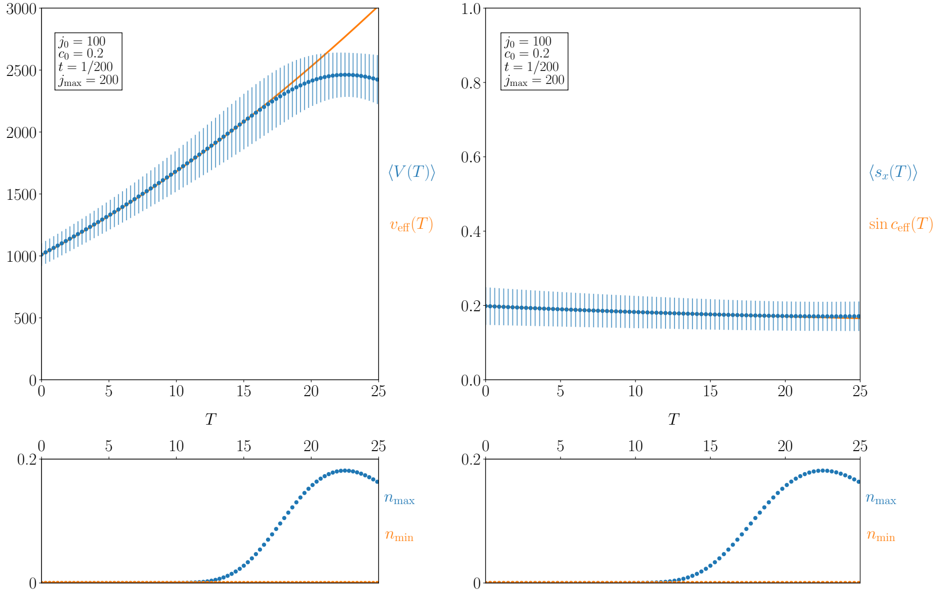
The upper limit $j_{\max}$ is introduced artificially, "by hand". In the calculations that follow, we use $j_{\max} = 200$, giving a Hilbert space of dimension $\dim \mathcal{H} = 8 \times 10^6$.

We then evaluate $|\psi(T)\rangle = e^{-i\hat{H}_{\text{phys}}T}|\psi_0\rangle$ using numerical routines (`expm_multiply` from `scipy.sparse.linalg` and `expv` from `ExponentialUtilities.jl`) that compute the product $e^A v$ for a sparse matrix A and vector v.

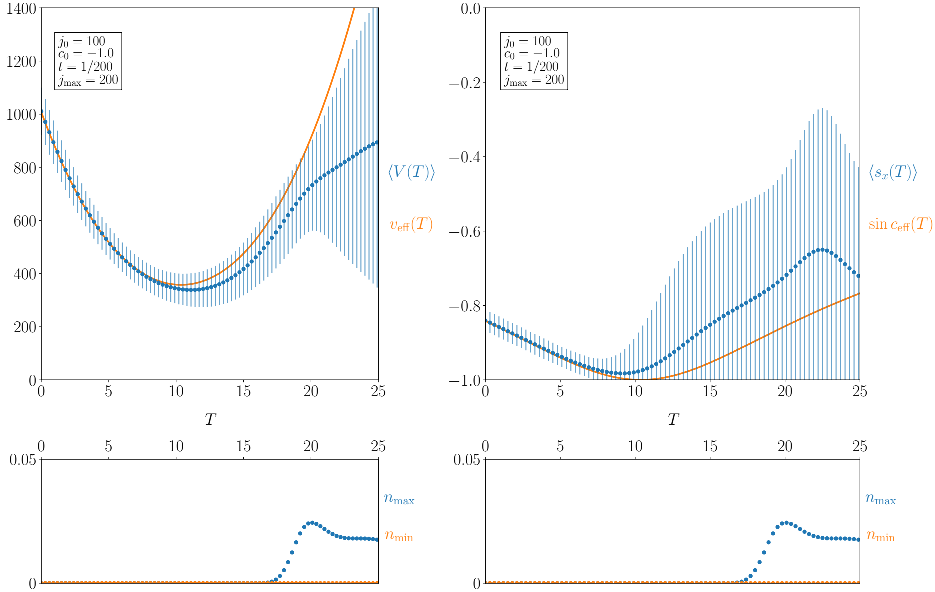
To assess the range of validity of the numerical simulation, we monitor the "occupation number" of the basis states in which any spin is maximal:

$$n_{\max} = \sum_{\text{any } j_a = j_{\max}} |c_{j_x j_y j_z}|^2 \qquad\qquad |\psi(T)\rangle = \sum_{j_x j_y j_z} c_{j_x j_y j_z}(T) |j_x j_y j_z\rangle$$

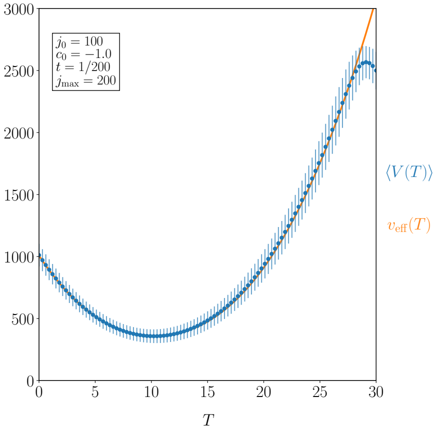
Euclidean model: Expanding universe

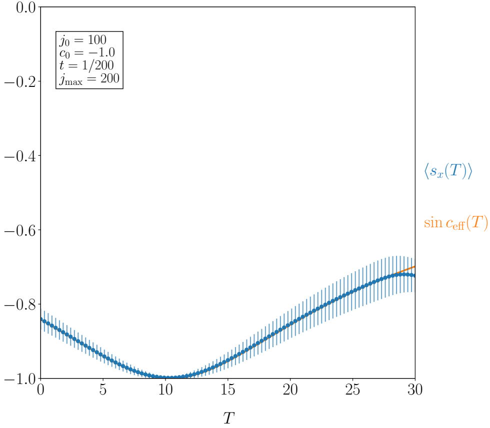


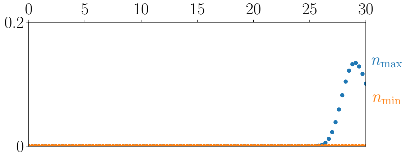
Euclidean model: Contracting universe



Toy model with no factors of $1/\sqrt{\hat{j}_i}$ in the Hamiltonian

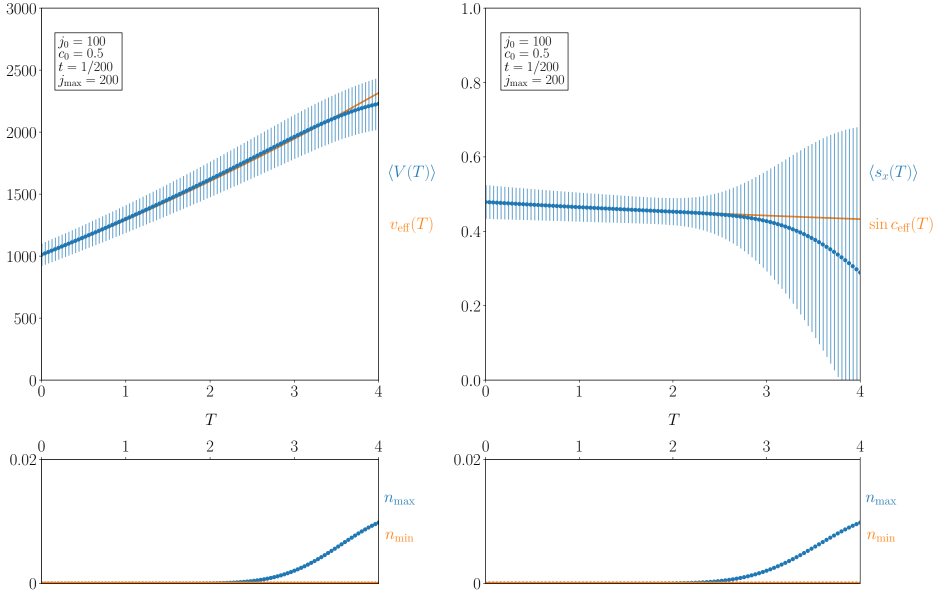




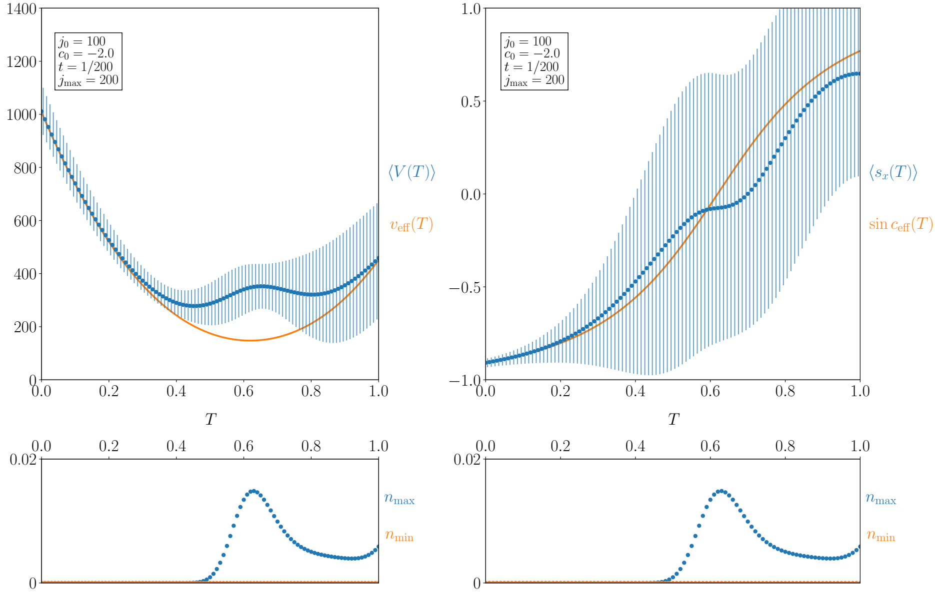


$$\hat{H}_{\text{phys}} = -\sqrt{\hat{j}_x}\,\hat{s}_y^{(1)}\hat{s}_z^{(1)} + \text{cycl. perm.}$$

Lorentzian model: Expanding universe



Lorentzian model: Contracting universe



Summary

We reviewed the construction of an operator representing the scalar curvature in loop quantum gravity restricted to a fixed cubic graph. The operator is obtained as a direct quantization of the Ricci scalar as a function of the Ashtekar variables.

Based on a formal similarity between the Hamiltonian in loop quantum cosmology and in the one-vertex model of quantum-reduced loop gravity, we proposed a possible new expression for the Lorentzian part of the Hamiltonian in LQC. The scalar curvature is represented by a non-trivial polymerized expression, which nevertheless goes to zero in the limit $\mu \to 0$.

We showed that the time evolution of one-vertex states in quantum-reduced loop gravity is accessible via numerical computations (over short enough time intervals, due to the cutoff imposed on the spin quantum numbers).

For an initially contracting universe, the quantum dynamics of semiclassical states around the bounce seems to match the semiclassical effective dynamics rather poorly, especially if the Lorentzian term is included in the Hamiltonian.

Thank you for your attention