Intrinsic rigidity of extremal horizons and black hole uniqueness

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Outline

- Black holes & extremal horizons

  Note: first uniqueness theorem for extremal horizons by Jurek [Lewandowski, Pawlowski '02]

- Intrinsic rigidity theorem for extremal horizons [Dunajski, JL '23]

  Einstein-Maxwell theory [Colling, Katona, JL '24]

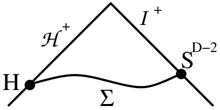
  See also talk by Colling.

- Applications to black hole uniqueness theorems [Katona, JL '23]

Black holes in General Relativity

Uniqueness theorem [Israel, Carter, Hawking, Robinson, Mazur, Bunting, '70...]

The DOC of $d = 4$ asymptotically flat, stationary, vacuum, analytic, spacetime with connected black hole is a Kerr solution.



- Topology [Hawking '72; Chrusciel, Wald '94]: Cross-sections $H \cong S^2$.
  Rigidity [Hawking '72]: stationary $\implies$ axisymmetric or static
- $\mathcal{H}^+$ is Killing horizon: $\nabla_V V = \kappa V$ on $\mathcal{H}^+$, normal Killing V

  Extremal $\kappa = 0$: [Meinel et al '08; Figueras, JL '09; Chrusciel, Nguyen '10]
  Requires *near-horizon geometry*...

Extremal horizons

- $\mathcal{H}^+$: null hypersurface with normal Killing vector V and $\kappa = 0$
  S: *cross-section*, $n := d - 2$-dimensional, transverse to V

- *Intrinsic data* on S: Riemannian (induced) metric g, 1-form X defined by $\nabla_W V = -\frac{1}{2}X(W)V$ on $\mathcal{H}^+$ for $W \in T\mathcal{H}^+$.

- $\mathrm{Ric}(\mathbf{g}) = \Lambda \mathbf{g}$ on $\mathcal{H}^+ \iff$ to *quasi-Einstein equation* on S:

$$\mathrm{Ric}(g) = \frac{1}{2}X \otimes X - \frac{1}{2}\mathcal{L}_X g + \Lambda g$$

  Note: intrinsic data decouples from extrinsic data iff $\kappa = 0$!

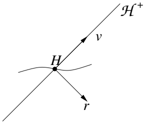
- Extremal isolated horizons also satisfy quasi-Einstein equation
  [Ashtekar, Beetle, Lewandowski, Pawlowski '02...]

Near-horizon geometry

- Isenberg-Moncrief (Gaussian-null) coords near an extremal $\mathcal{H}^+$

  $$\mathbf{g} = r^2 F(r,x)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + 2r\, X_a(r,x)\mathrm{d}v\mathrm{d}x^a + g_{ab}(r,x)\mathrm{d}x^a\mathrm{d}x^b$$

  $V = \partial_v$, ∂_r transverse, $\mathcal{H}^+ = \{r = 0\}$, x^a coords on S



- *Near-horizon geometry* [Reall '02]: associated spacetime which encodes only intrinsic data $(g, X, F)_{r=0}$ (scaling limit) .
  Einstein iff (S, g, X) quasi-Einstein, $F = \frac{1}{2}|X|^2 - \frac{1}{2}\mathrm{div}_g X + \Lambda$.
  [Chrusciel, Reall, Tod '05; Kunduri, JL '08...]
- Classify compact quasi-Einstein manifolds (S, g, X)? Difficult!

Static horizons

Static rigidity [Chrusciel, Tod, Reall '05; Bahuaud, Gunasekaran, Kunduri, Woolgar '22; Wylie '23; Kaminski, Lewandowski '24]

Let (S, g, X) be a compact quasi-Einstein manifold with $\mathrm{d}X = 0$:

- $\Lambda = 0$ then $X \equiv 0$ and $\mathrm{Ric}(g) = 0$ (trivial).
- $\Lambda < 0$ then either:
  - $X \equiv 0$ and $\mathrm{Ric} = \Lambda g$.
  - S is quotient of $\mathbb{R} \times N$, $g = \mathrm{d}z^2 + \tilde{g}$, $X \propto \partial_z$, $(N, \tilde{g})$, $\mathrm{Ric}_{\tilde{g}} = \Lambda \tilde{g}$.
- $\Lambda > 0$: $n = 2$ then $X \equiv 0$, $\mathrm{Ric} = \Lambda g$. $n \geq 3$ then X gradient.

- Near-horizon geometry static $V \wedge \mathrm{d}V = 0 \iff \mathrm{d}X = 0$ on S.
- $n \geq 3$, $\Lambda > 0$: classify nontrivial compact gradient solutions?

Axially symmetric two-horizons

Horizon uniqueness [Hajicek '74; Lewandowski, Pawlowski '02; Kunduri, JL '08]

Any axially-symmetric vacuum extremal horizon with an S^2 cross-section is an extremal Kerr or Kerr-(A)dS horizon.

- Axial symmetry reduces horizon equation to ODE system and general *local* solution can be found even for $\Lambda \neq 0$.
- Global analysis: metric that extends smoothly on S^2 ($\Lambda = 0$)

$$g = \frac{a^2(1+x^2)}{1-x^2}\mathrm{d}x^2 + \frac{4a^2(1-x^2)}{1+x^2}\mathrm{d}\phi^2 ,$$

$$X = \Gamma^{-1}K^\flat - \frac{\mathrm{d}\Gamma}{\Gamma}, \qquad K = \partial_\phi, \quad \Gamma = a^2(1+x^2)$$

$|x| < 1$, ∂_ϕ is 2π-periodic Killing field, $a > 0$ constant

Two-horizons

- General solution to horizon quasi-Einstein equation on S^2?
  - Axial symmetry motivated by rigidity theorem for black holes. But is there an intrinsic proof of axial symmetry?
  - Rigidity of linearised perturbs of the extremal Kerr horizon [Jezierski, Kaminski '12; Chrusciel, Szybka, Tod '17; Bauhaud, Gunasekaran, Kunduri, Woolgar '23]
- Intrinsic topology theorem: $R_g = \frac{1}{2}|X|^2 - \mathrm{div}_g X + n\Lambda$. $\Lambda \geq 0$: $\int_S R_g \geq 0$ so $S = S^2, T^2$. T^2 case is trivial $X \equiv 0$.
- $\Lambda < 0$: genus $g > 0$ horizons all trivial $X \equiv 0$, $\mathrm{Ric}(g) = \Lambda g$ [Dobkowski-Rylko, Kaminski, Lewandowski, Szereszewski '18]

Intrinsic rigidity of extremal horizons

Rigidity theorem [Dunajski, JL '23]

Let (S, g, X) be an n-dimensional compact Riemannian manifold with non-gradient X that satisfies horizon quasi-Einstein equation. Then there exists a Killing vector field K such that $[K, X] = 0$.

- $\mathrm{d}X \neq 0 \iff$ non-static. Complements static classification.
- More rigid than Einstein manifolds!
- $[K, X] = 0 \implies K$ a Killing vector of near-horizon geometry.

  $\Lambda > 0, n > 2$ proof more nontrivial [Colling, Dunajski, Kunduri, JL '24]
  Unified proof for all Λ [Kaminski, Lewandowski '24]

Corollaries

Horizon uniqueness theorem [Dunajski, JL '23]

The extremal Kerr horizon, possibly with cosmological constant, is unique solution to horizon quasi-Einstein eq on S^2.

- This completes classification of vacuum extremal horizons with a compact cross-section for any Λ.

Near-horizon symmetry enhancement theorem [Dunajski, JL '23]

A vacuum near-horizon geometry with compact cross-sections has an isometry group of a 2d maximally symmetric space (3d orbits).

- This is a significant generalisation of prior NH-symmetry theorem which *assumes* $U(1)^{n-1}$ symmetry [Kunduri, JL, Reall '07]

Rigidity proof - strategy

- Inspired by Kerr solution, for any function $\Gamma > 0$, let

$$K := \Gamma X + d\Gamma$$

  Idea: try to prove K is a Killing field for suitable choice of Γ.

- Given a 1-form X on a compact Riemannian mfd (S, g) there exists a smooth $\Gamma > 0$ so $\text{div}_g K = 0$. [Tod '92; JL, Reall '13]
  - Elliptic operator $L\psi := -\text{div}_g(d\psi + X\psi)$ on (S, g). Principal eigenfunction exists and is positive $L\psi = \mu\psi$.
  - Integrate over S implies $\mu = 0$. Take Γ to be solution $L\Gamma = 0$.

Rigidity proof – remarkable identity

- Precise form of quasi-Einstein equation implies, for *all* $\Gamma > 0$,

$$\begin{aligned}
\tfrac{1}{4}|\mathcal{L}_K g|^2 &= \mathrm{div}_g Y + Z\mathrm{div}_g K \\
Y &:= \tfrac{1}{2}(\mathcal{L}_K g)(K,\cdot) - \tfrac{1}{2}K\Delta\Gamma - \tfrac{1}{2}K\mathrm{div}_g K - \Lambda\Gamma K \\
Z &:= -\frac{1}{2\Gamma}|K|^2 + \tfrac{1}{2}\Delta\Gamma + \tfrac{1}{2}\mathrm{div}_g K + \frac{1}{2\Gamma}\mathcal{L}_K\Gamma + \Lambda\Gamma
\end{aligned}$$

- Now take Γ to be principal eigenfunction so $\mathrm{div}_g K = 0$:

$$\int_S |\mathcal{L}_K g|^2 \mathrm{dvol}_g = 0 \qquad \Longrightarrow \qquad \mathcal{L}_K g = 0$$

- Identity reduces to $\mathrm{div}_g Y = 0 \iff \Delta(\mathcal{L}_K\Gamma) = -2\Lambda\mathcal{L}_K\Gamma$; $\Lambda \le 0 \implies \mathcal{L}_K\Gamma = 0$ so $[K, X] = 0$. $\Lambda > 0$ a bit more work.

  Recent generalised identity that allows unified proof for all Λ.

  [Kaminski, Lewandowski '24]

Electrovacuum extremal horizons

Rigidity theorem [Colling, Katona, JL '24]

Let (S, g) be a 2-dimensional compact Riemannian mfd that satisfies quasi-Einstein-Maxwell eqs with non-gradient X. Then there exists a Killing vector field K and $[K, X] = \mathcal{L}_K \psi = \mathcal{L}_K \beta = 0$.

- Einstein-Maxwell eqs restrict to quasi-Einstein-Maxwell eqs on cross-section S. Maxwell data: EM potentials ψ, β
- Remarkable identity generalises: if $K = \Gamma X + \mathrm{d}\Gamma$ for any $\Gamma > 0$

$$\frac{1}{4}|\mathcal{L}_K g|^2 + |\nabla(\Gamma\sqrt{\psi^2 + \beta^2})|^2 = \mathrm{div}_g \tilde{Y} + \tilde{Z}\mathrm{div}_g K$$

- S^2 and axial sym $\implies$ extremal Kerr-Newman$-\Lambda$ horizon [Lewandowski, Pawlowski '02; Kunduri, JL '08]

  Completes classification (higher genus can be ruled out)

Transverse deformations of extremal horizons

- Given a near-horizon geometry a corresponding black hole solution may not exist or be unique.

- *Transverse derivatives* at extremal horizon: $g_{ab}^{(n)} := \partial_r^n g_{ab}|_{r=0}$. Einstein $\implies$ elliptic problem on S at each order n [Li, JL '15]

- Vacuum axisymmetric $n = 1$ transverse derivatives of extremal Kerr(-Λ) horizon are unique. [Li, JL '15 '18]

- Unique at every order? Don't expect so, no asymptotic input!

  E.g. $n = 1$ derivatives of Reissner-Nordstrom/Kerr-Newman not unique [Li, JL '18, Kolanowski '21]

Example: extremal Schwarzschild de Sitter

Uniqueness theorem [Katona, JL '23]

Any analytic Einstein spacetime with $\Lambda > 0$, that contains a static extremal Killing horizon with a compact cross-section S (if $n > 2$ maximal sym.), is locally isometric to extremal Schwarzschild dS spacetime or its near-horizon geometry $\mathrm{dS}_2 \times S^n$ (Nariai solution).

- $d = 4$: static horizon with compact cross-section must have maximal symmetry [Chrusciel, Reall, Tod '05].
- No global assumptions such as asymptotics or topology! Hence rules out (analytic) multi-black holes.
- First uniqueness theorem for extremal vacuum black holes with Λ for $d \geq 4$ dimensions.

Proof sketch

- Horizon data: $g^{(0)} = r_0^2 d\Omega_{d-2}^2$, $X_a^{(0)} = 0$, $F^{(0)} = \Lambda$ (Nariai)
- Gauge freedom $g_{ab}^{(1)} \to g_{ab}^{(1)} + \nabla_a \nabla_b f$, fix trace $g_{ab}^{(1)}$ to const. First order $g_{ab}^{(1)} = C g_{ab}^{(0)}$, $X_a^{(1)} = 0$, $F^{(1)} = \text{const}$ [Li, JL, '18]
- Einstein equation: assume soln to order $n-1$, then for $n \geq 2$

$$-\nabla^2 \hat{g}_{ab}^{(n)} = -\Lambda \left(n^2 + n + \frac{2}{d-3} \right) \hat{g}_{ab}^{(n)} \qquad \text{on } S$$

where $\hat{g}_{ab}^{(n)} := g_{ab}^{(n)} - \frac{C^2}{2} \delta_{n,2} g_{ab}^{(0)}$ tracefree.
$\Lambda > 0 \implies \hat{g}_{ab}^{(n)} = 0$ as $-\nabla^2$ positive, $X_a^{(n)} = 0$, $F^{(n)} = \text{const}$

- Analytic $\implies g_{ab} = r_0^2 \left(1 + \frac{Cr}{2}\right)^2 d\Omega_{d-2}^2$, $X_a = 0$, $F = F(r)$. $C \neq 0$ is Schwarzschild-dS, $C = 0$ is near-horizon geometry

Comments

- Constructive uniqueness proof for spacetimes with prescribed near-horizon geometry.
- $\Lambda < 0$ extremal hyperbolic Schwarzschild-AdS: uniqueness $\lambda_n = (n^2 + n - 2)|\Lambda| \notin$ spec of Laplacian on hyperbolic surface and no cohomological obstruction to first order uniqueness.
- Generalises to Einstein-Maxwell theory if $A_H \Lambda \geq 2\pi$ [Katona '24]
- Argument that generic extremal black holes with $\Lambda \neq 0$ do not have smooth horizons [Horowitz, Kolanowski, Santos '22]

Summary & Outlook

- Geometry of extremal black holes is a quasi-Einstein manifold. Study horizon topology/geometry independently to black hole.
- Extremal horizons enjoy remarkable intrinsic rigidity. Even for $\Lambda \neq 0$ (no BH no-hair theorems here!).
  - Intrinsic proof of uniqueness of extremal Kerr horizon!
  - Axially symmetric in all dimensions!
  - Near-horizon symmetry enhancement generic
- Novel uniqueness theorem for analytic $\Lambda > 0$ spacetimes containing a *static* extremal horizon. Persists with rotation?