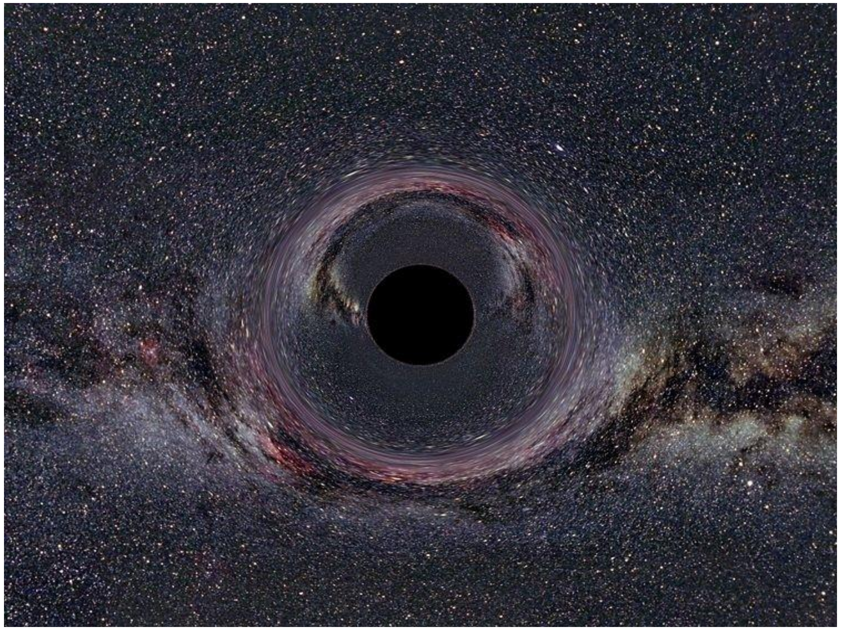
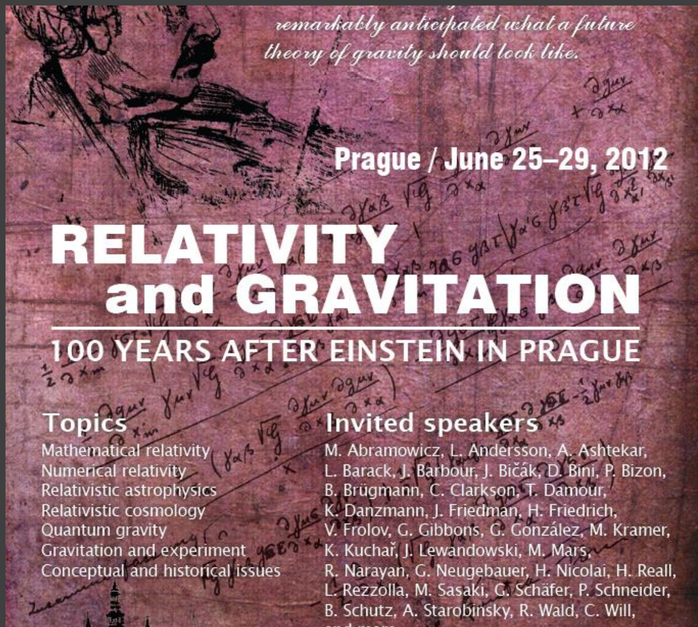
On a lower dimensional Killing vector origin of irreducible Killing tensors

David Kubizňák
(Institute of Theoretical Physics, Charles University)



Geometry of classical and quantum spacetimes – Jerzy Lewandowski memorial conference
Warszawa, Poland, Sep 15-19, 2025


remarkably anticipated what a future theory of gravity should look like.
Prague / June 25–29, 2012
RELATIVITY and GRAVITATION
100 YEARS AFTER EINSTEIN IN PRAGUE
Topics
Mathematical relativity
Numerical relativity
Relativistic astrophysics
Relativistic cosmology
Quantum gravity
Gravitation and experiment
Conceptual and historical issues
Invited speakers
M. Abramowicz, L. Andersson, A. Ashtekar,
L. Barack, J. Barbour, J. Bičák, D. Bini, P. Bizon,
B. Brügmann, C. Clarkson, T. Damour,
K. Danzmann, J. Friedman, H. Friedrich,
V. Frolov, G. Gibbons, G. González, M. Kramer,
K. Kuchař, J. Lewandowski, M. Mars,
R. Narayan, G. Neugebauer, H. Nicolai, H. Reall,
L. Rezzolla, M. Sasaki, G. Schäfer, P. Schneider,
B. Schutz, A. Starobinsky, R. Wald, C. Will,
and more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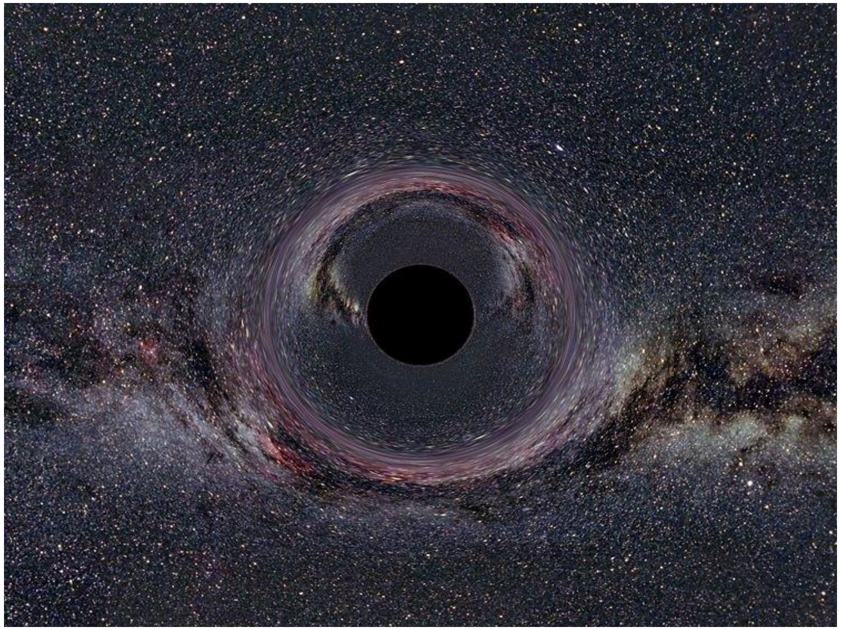

Geometry of classical and quantum spacetimes – Jerzy Lewandowski memorial conference
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Plan of the talk

1) What are hidden (dynamical) symmetries?
2) More on Killing tensors
3) Generalized Lense-Thirring metrics
4) Non-commutativity of isometries gives rise to Killing tensors
5) Summary

Based on:

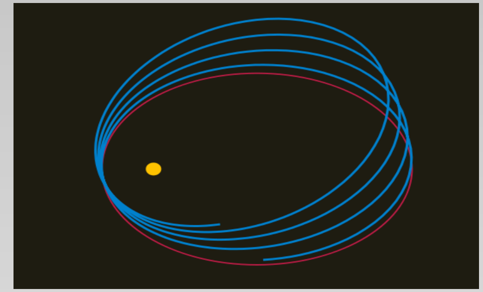
- F. Gray, DK, *Slowly rotating black holes with exact Killing tensor symmetries*, PRD 105 (2022) 6, 064017.
- F. Gray, G. Odak, P. Krtous, DK, *On a lower-dimensional Killing vector origin of irreducible Killing tensors*, JHEP 07 (2025) 098.

1) What are hidden (dynamical) symmetries?

Laplace-Runge-Lenz vector

Central force:

$$E\ ,\ \vec{L}$$

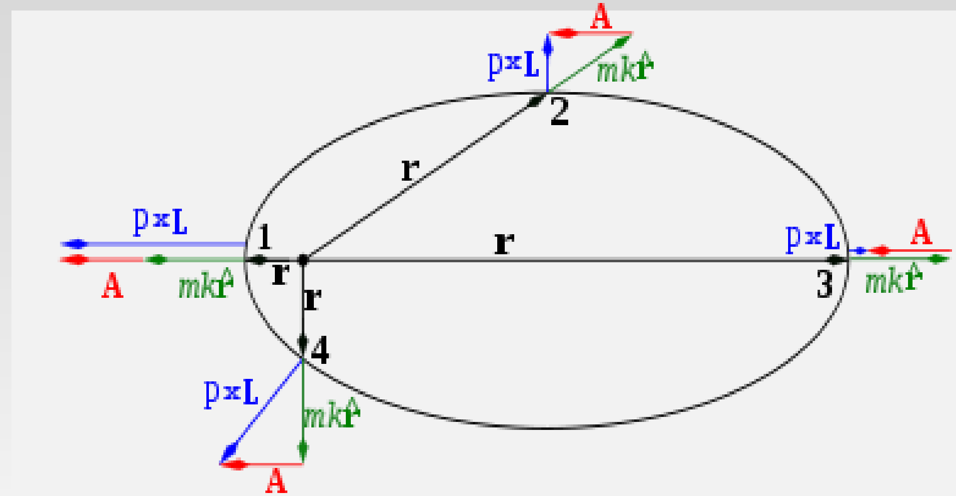


Kepler problem:

$$\vec{F} = -\frac{k}{r^2}\hat{r}$$

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$$

Laplace-Runge-Lenz vector



Wikipedia

- motion maximally superintegrable

$$\vec{A} \cdot \vec{L} = 0 \qquad A^2 = m^2k^2 + 2mEL^2$$

- Example of a **hidden (dynamical) symmetry**

Hamiltonian dynamics

Symplectic manifold: $$\omega = \frac{1}{2}\omega_{AB}\, d\xi^A \wedge d\xi^B$$ (non-degenerate, closed 2-form)

Hamiltonian vector flow generated by function f: $$X_f^A = \omega^{AB}\partial_B f$$

Darboux coordinates: $\xi^A = (x^\mu, p_\nu)$ s.t $\omega = dp_\mu \wedge dx^\mu$

Noether's theorem (phase space)

Let Hamiltonian H preserved by an infinit. transf. $\delta x^\mu \, , \;\; \delta p_\nu$
Then, there exists a **conserved quantity Q**:

$$\{Q, H\} = 0$$

$$X_Q = \delta x^\mu \frac{\partial}{\partial x^\mu} + \delta p_\nu \frac{\partial}{\partial p_\nu}$$

Hidden (dynamical) symmetries

Spec: **Phase space** is a cotangent bundle of manifold M, T*(M).

Then there exists a **canonical projection:**

$$\pi \; : \; T^*(\mathcal{M}) \rightarrow \mathcal{M}$$

Can distinguish isometries from dynamical symmetries:

$$\pi^*(X_Q) = \begin{cases} \text{vector field on } M & \underline{\textit{isometry}} \\ \text{not well defined on } M & \underline{\textit{dynamical symmetry}} \end{cases}$$

Laplace-Runge-Lenz:

$$X_{A^i} = \left(2x^i p^k - \delta^i_k x \cdot p - p^i x^k\right) \frac{\partial}{\partial x^k} - \left(\delta^i_k p^2 - p^i p_k - mk\delta^i_k \frac{1}{r} + mk \frac{x^i x^k}{r^3}\right) \frac{\partial}{\partial p^k}$$

$$\pi^*(X_{A^i}) = \left(2x^i p^k - \delta^i_k x \cdot p - p^i x^k\right) \frac{\partial}{\partial x^k}$$

dynamical symmetry

Symmmetries in GR

Particle motion

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

geodesics:

$$p^\mu \nabla_\mu p^\nu = 0$$

a) Linear in momentum conserved quantities:

$$C_K = K^\mu p_\mu \quad \Leftrightarrow \quad \nabla_{(\mu} K_{\nu)} = 0$$

...Killing vector equation

Proof:

$$\dot{C}_K = p^\nu \nabla_\nu (K^\mu p_\mu) = p^\nu p^\mu \nabla_{(\nu} K_{\mu)} + K^\mu \underbrace{p^\nu \nabla_\nu p_\mu}_{0} = 0$$

Hamiltonian vector field:

$$X_{C_K} = K^\mu \frac{\partial}{\partial x^\mu} - \frac{\partial K^\lambda}{\partial x^\mu} p_\lambda \frac{\partial}{\partial p_\mu}$$

$$\pi_*(X_{C_K}) = K^\mu \frac{\partial}{\partial x^\mu} = K$$

...isometry

b) Higher-order conserved quantities

$$C_K = K^{\mu_1 \dots \mu_p} p_{\mu_1} \dots p_{\mu_p}$$

$$\nabla_{(\mu} K_{\nu_1 \dots \nu_p)} = 0$$

...Killing tensor equation

Walker & Penrose, Comm. Math. Phys. 18 , 265 (1970). (Stackel 1895).

$$\pi_*(X_{C_K}) = pK^{\mu_1 \dots \mu_{p-1} \nu} p_{\mu_1} \dots p_{\mu_{p-1}} \frac{\partial}{\partial x^\nu}$$

...dynamical symmetry

Symmetries in GR

Explicit symmetries

…Killing vectors (isometries)

Hidden symmetries

…symmetric **Killing tensors** (dynamical symmetries)

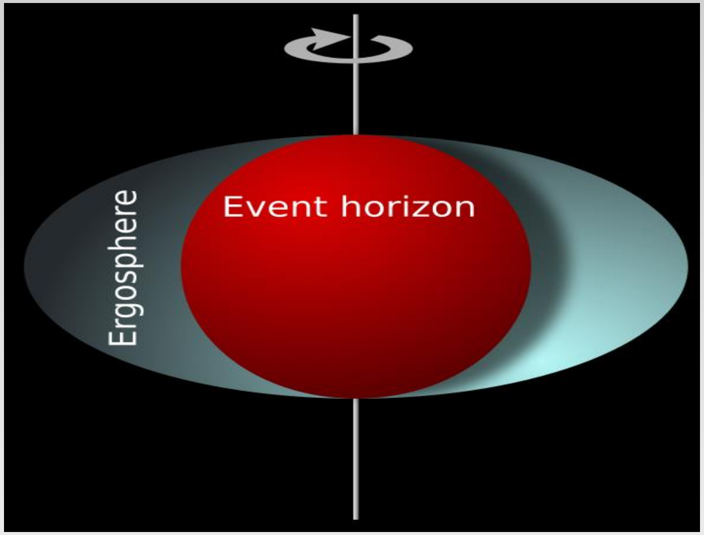
…Killing-Yano tensors (even more "fundamental" – they square to Killing tensors)

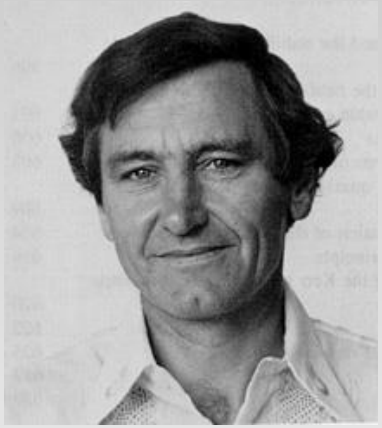
$$K_{\mu\nu} = f_{\mu\alpha} f_{\nu}{}^{\alpha}$$

Although we derived these as symmetries of the particle motion, they have far-reaching consequences for the **properties of the spacetime** and the dynamics of fields in it.

Famous example: Kerr geometry

- **Unique vacuum** solution of Einstein equations describing a rotating black hole in 4d





Roy Patrick Kerr

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - a dt\right]^2$$

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 f + a^2 \qquad f = 1 - \frac{2M}{r}$$

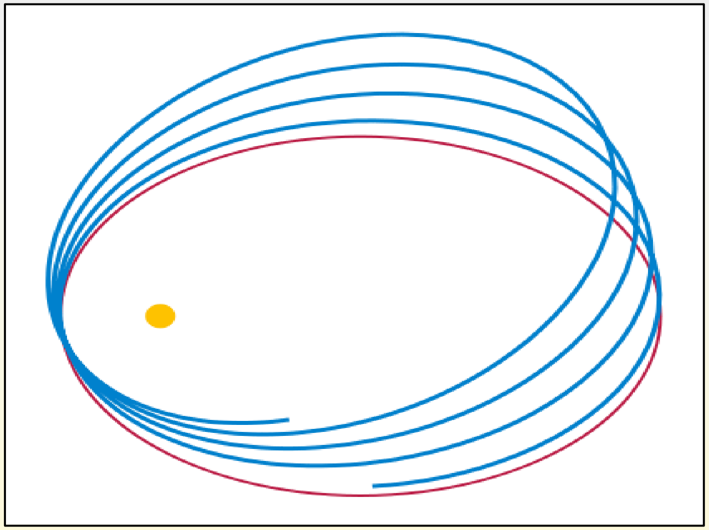
Remarkable properties

- **Geodesic motion is completely integrable**

  - The metric is stationary and axisymmetric: $\boldsymbol{k} = \partial_t \,, \ \boldsymbol{m} = \partial_\varphi$

$$g_{ab}u^a u^b = -1 \,, \quad k_a u^a = -E \,, \quad m_a u^a = L$$

- **1968 Carter** discovers an additional constant of motion which results in a complete integrability of geodesic motion



$$K_{ab}u^a u^b = K$$

- **Field equations decouple and separate**

  Scalar field, Dirac, electromagnetic, and gravitational perturbations **decouple and separate variables** (Carter 1968, Teukolsky 1972, Chandrasekhar & Page 1976, Wald 1978)

Enables to study:

- black hole shadow
- plasma accretion
- black hole stability
- Hawking evaporation
- …

- **Kerr-Schild form**: the metric can be written as a linear in mass deformation of the flat space

$$\boldsymbol{g} = \mathring{\boldsymbol{g}} + \frac{2Mr}{\Sigma}\,\boldsymbol{l}\,\boldsymbol{l}$$

- **Special algebraic type** of the Weyl tensor

2) More on Killing tensors

More on Killing tensors

=totally symmetric tensors obeying

$$\nabla^{(a_1} K^{a_2 a_3 \ldots a_{p+1})} = 0.$$

Generate ***constants of geodesic motion*** of degree p

M. Walker and R. Penrose, Comm. Math. Phys. 18 , 265 (1970).

$$\mathcal{K}_p = K^{a_1 \ldots a_p} \, p_{a_1} \cdots p_{a_p}$$

Poisson commute with the Hamiltonian generating geodesic flow

$$\mathcal{H} = \frac{1}{2} g^{ab} p_a p_b$$

Reducibility

$$K_{(1)}^{(a} K_{(2)}^{bc)}, \quad \text{or} \quad K_{(3)}^{(a} K_{(4)}^{b} K_{(5)}^{c)},$$

Algebra of Killing tensors

Killing tensors form an algebra with respect to (symmetric) ***Schouten-Nijenhuis brackets***:

$$\{\mathcal{K}_p, \mathcal{K}_q\} = \frac{\partial \mathcal{K}_p}{\partial q^i}\frac{\partial \mathcal{K}_q}{\partial p_i} - \frac{\partial \mathcal{K}_q}{\partial q^i}\frac{\partial \mathcal{K}_p}{\partial p_i}$$
$$\equiv [K_p, K_q]^{a_1 a_2 \ldots a_{p+q-1}}_{\mathrm{SN}} p_{a_1} p_{a_2} \cdots p_{a_{p+q-1}}.$$

$$[K_p, K_q]^{a_1 \ldots a_{p+q-1}}_{\mathrm{SN}} = p K_p{}^{c(a_1 \ldots p-1} \nabla_c K_q{}^{a_p \ldots a_{p+q-1})} - q K_q{}^{c(a_1 \ldots a_{q-1}} \nabla_c K_p{}^{a_q \ldots a_{q+p-1})}$$

- <u>Note also</u> that in principle one can generate higher-rank Killing tensors by employing SN brackets.

$$[K_p, K_q]^{a_1 \ldots a_{p+q-1}}_{\mathrm{SN}}$$

Algebra of Killing tensors

- Spec: **metric** g is a (trivial) Killing tensor

$$[\xi, g]_{\rm SN}^{ab} = \mathcal{L}_\xi g^{ab} = -2\nabla^{(a}\xi^{b)}$$

$$[K_p, g]_{\rm SN}^{a_1\dots a_p} = -p\nabla^{(a_1}K_p^{a_2\dots a_p)}$$

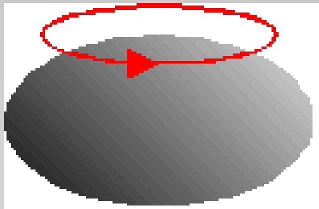
- In other words: **Killing vector** and **Killing tensor** equations can be conveniently expressed as

$$[\xi, g]_{\rm SN} = 0 \qquad [K_p, g]_{\rm SN} = 0$$

Examples of spacetimes with rank-2 KTs

1. Kerr geometry (in all dimensions)

   P. Krtouš, D. Kubizňák, D. N. Page, and V. P. Frolov, Killing-Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions, JHEP 0702 (2007) 004



2. Taub-NUT space: generalization of Runge-Lenz vector

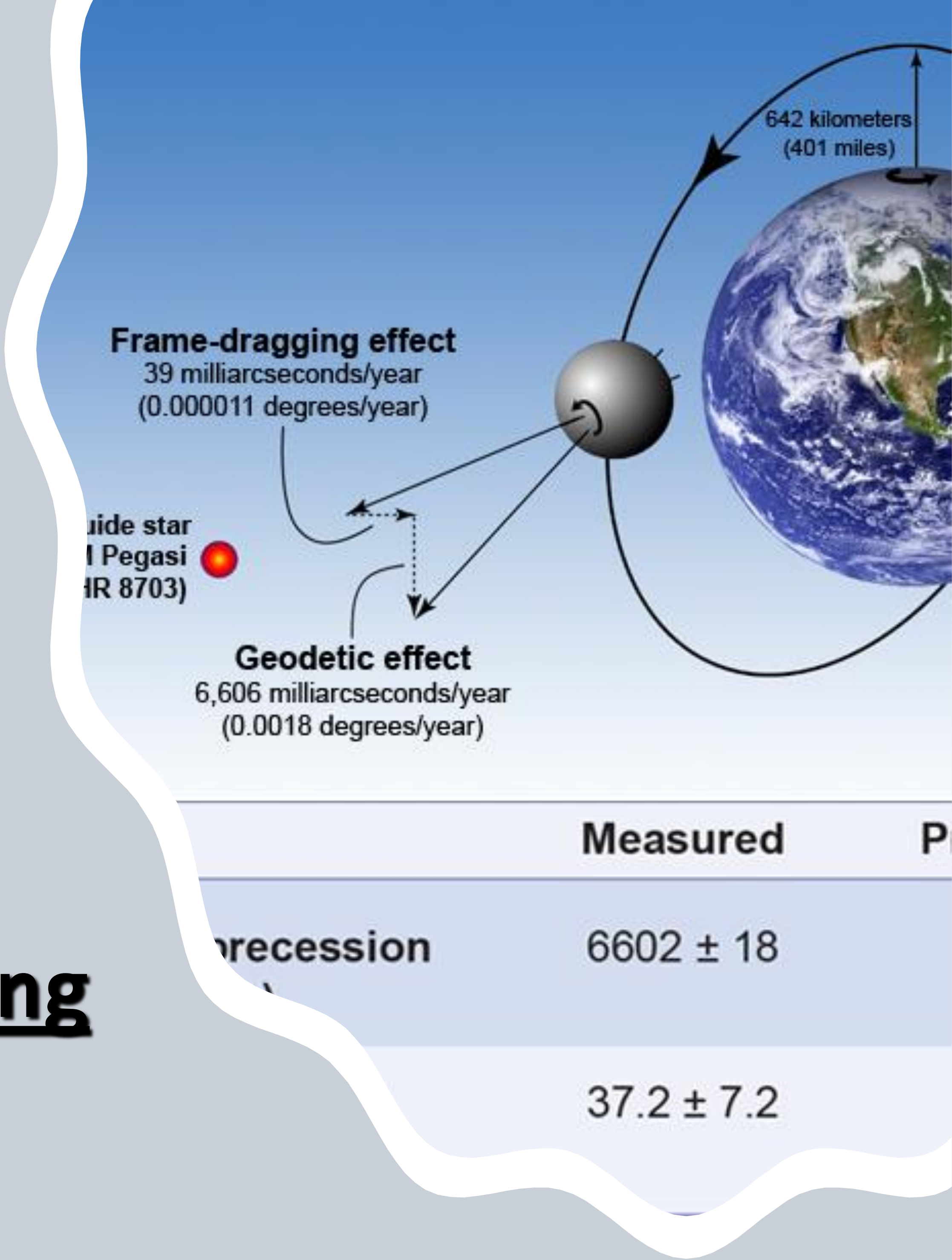
   G. W. Gibbons and P. J. Ruback, "The Hidden Symmetries Of Taub-NUT And Monopole Scattering," Phys. Lett. B 188 (1987) 226.

3. Various SUGRA black holes

   D.D. Chow, Symmetries of supergravity black holes, Class. Quant. Grav. 27, 205009 (2010) , arXiv:0811:1264.

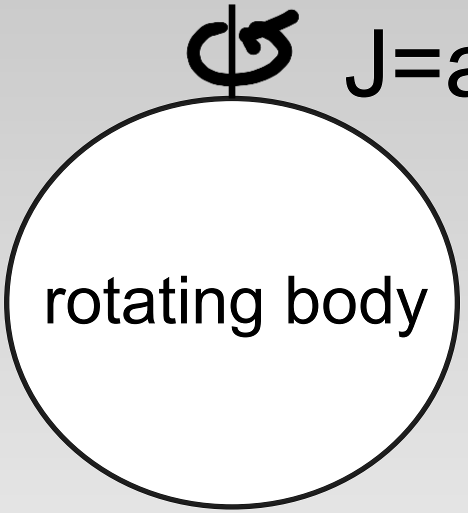
What about spacetimes with irreducible higher-rank Killing-Stackel tensors?

- G. Gibbons, T. Houri, DK, C. Warnick, *Some spacetimes with higher-rank Killing tensors*, PLB700 (2011), 68.



3) Generalized Lense-Thirring metrics

Lense-Thirring spacetime (1918)



J=aM

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta\, dt\, d\phi + r^2(\sin^2\theta\, d\phi^2 + d\theta^2)\,, \qquad f = 1 - \frac{2M}{r}$$

- Spacetime outside a **slowly rotating body**
- **Approximate** (linear in a) vacuum solution of EE
- Linear in a approximation to **Kerr** (1963)
- Encodes **gravitomagnetic effects** (Gravity **Probe B**)

Let us complete the square

- Instead of $$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta\, dt\, d\phi + r^2(\sin^2\theta\, d\phi^2 + d\theta^2)\,, \quad f = 1 - \frac{2M}{r}$$

- **Consider** the following **"exact spacetime"**:

$$ds^2 = -Nf\,dt^2 + \frac{dr^2}{f} + r^2\sin^2\theta\left(d\phi + \frac{ap}{r^2}dt\right)^2 + r^2 d\theta^2$$

- Admits the **Painleive-Gullstrand (PG) form**

- Admits the **exact irreducible Killing tensor**

$$K = \frac{1}{\sin^2\theta}(\partial_\phi)^2 + (\partial_\theta)^2 \qquad \left(K = L_x^2 + L_y^2 + L_z^2 \right)$$

J. Baines, T. Berry, A.Simpson, M. Visser, Arxiv:2110.01814.

Upgraded LT spacetimes

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\phi_i + \frac{a_i p_i}{r^2} dt\Big)^2 + r^2 (\sum_{i=1}^{m} d\mu_i^2) + \epsilon r^2 d\nu^2$$

F. Gray, DK, *Slowly rotating black holes with exact Killing tensor symmetries, Phys. Rev. D105 (2022) 6, 064017;* ArXiv:2110:14671.

- Admits: horizon, ergoregion, PG form, (m+1) KVs, …
- Define: $$2b^{(I)} \equiv r^2 (dt + \sum_{i \in I} a_i \mu_i^2 d\phi_i)\,, \qquad h^{(I)} \equiv db^{(I)}$$

  (relevant order expansion of PKY of Myers-Perry)

- Construct $$f^{(I)} \equiv \frac{1}{(|I|+1)!} * (\underbrace{h^{(I)} \wedge \cdots \wedge h^{(I)}}_{|I|+1 \text{ times}})$$

$$K^{(I)}_{\mu\nu} = \Big(\prod_{i \in S} a_i\Big)^{-2} (f^{(I)} \cdot f^{(I)})_{\mu\nu}$$

…rank-2 KTs

Towers of hidden symmetries

- The **tower of irreducible KTs grows quadratically** with number of dimensions (compared to linear for MP)

- For high enough d, provides an example of a spacetime with **more hidden** than explicit symmetries

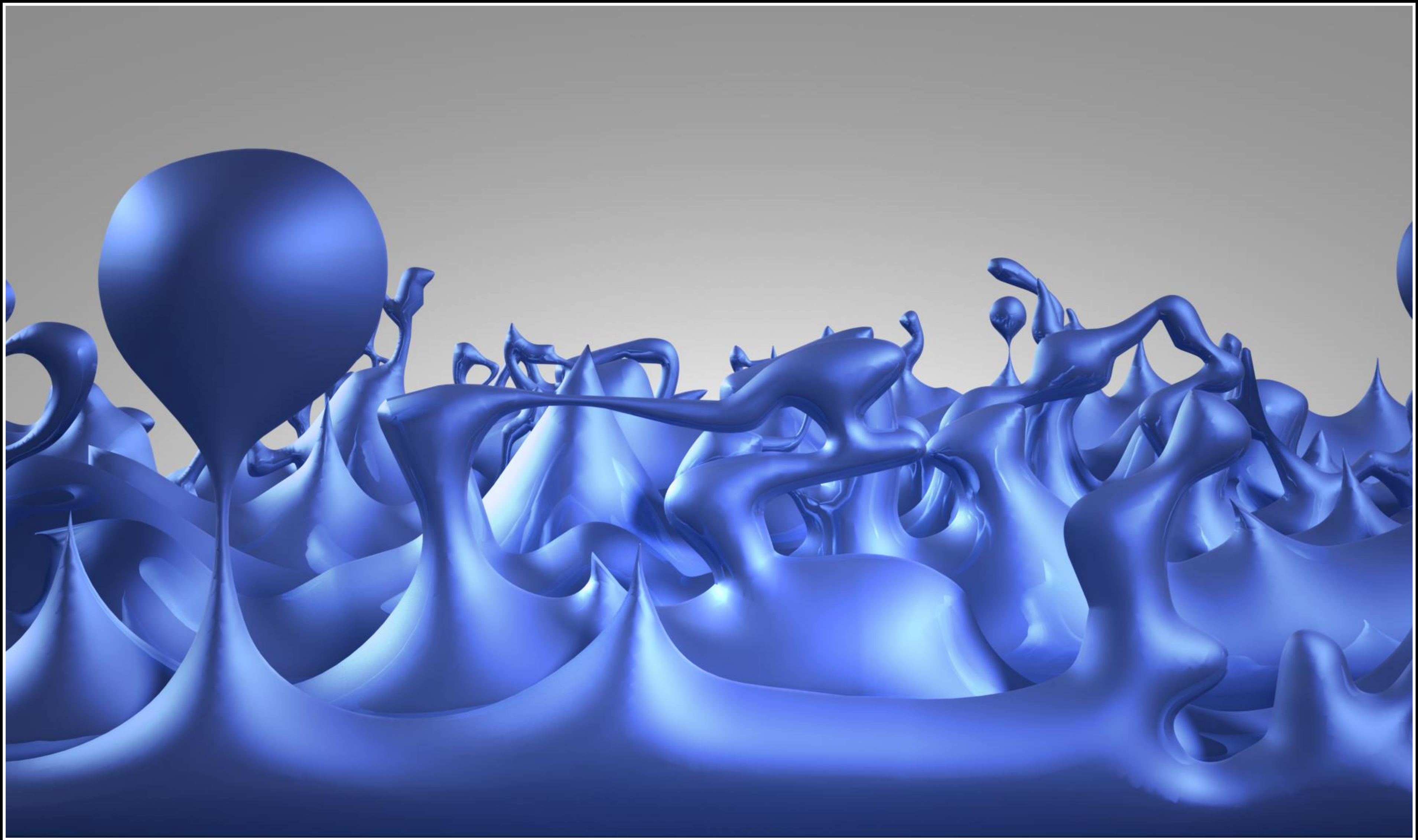
- **Higher-rank KTs:** (e.g. 6D)

$$[K^{(\emptyset)}, K^{(1)}]_{\mathrm{SN}} = 0 = [K^{(\emptyset)}, K^{(2)}]_{\mathrm{SN}}$$

$$M = [K^{(1)}, K^{(2)}]_{\mathrm{SN}}$$ … irreducible rank-3 KT

  - Further brackets with K1 and K2 generate rank 4 KT's, and so on …. does the tower **terminate** at some point?

- Example of a **physically motivated** spacetime with **higher rank Killing tensors**



4) Non-commutativity of isometries gives rise to Killing tensors

Key questions about the previous example

1) Where do the above Killing tensors **come from**? Why are they irreducible? And why is there so many?

2) Can we generate irreducible KT of arbitrary **high rank**? Does the tower terminate at some point?

3) Do we have an **on-shell example** of physical spacetime?

4) Are there **other metrics** like that?

Basic idea

- Consider a foliation by **codimension-2 hypersurfaces** $\mathcal{S}^{(r,t)}$

$$g = -Nf dt^2 + \frac{dr^2}{f} + \gamma_{AB}(dx^A + \nu^A dt)(dx^B + \nu^B dt)$$

where $N = N(t,r)$ $f = f(t,r)$

$$\gamma_{AB} = \gamma_{AB}(t,r,x^C)$$

- Consider a **natural lift** of objects on S:

$X^{A_1 \dots A_p} \in T^p\mathcal{S}$ $T^p\mathcal{M} \ni \hat{X} \equiv X^{A_1 \dots A_p}\partial_{A_1} \dots \partial_{A_p}$

$$\Rightarrow \quad [\hat{X}, \hat{Y}]_{\mathrm{SN}} = \widehat{[\![X, Y]\!]}_{\mathrm{SN}}$$

Basic idea

- To find a symmetry of the full spacetime we calculate

$$[\hat{X}, g^{-1}]^{\mu_1 \ldots \mu_{p+1}}_{\mathrm{SN}} = [\![X, \gamma^{-1}]\!]_{\mathrm{SN}}$$ **+ other terms**

- **Sufficient conditions:** $[\![X, \gamma^{-1}]\!]_{\mathrm{SN}} = 0\,.$

$$\partial_t X = 0\,, \quad \partial_r X = 0\,, \quad [\![X, \nu]\!]_{\mathrm{SN}} = 0$$

- Spec: Let ξ_1 be a **KV on S**. Assume further

$$\nu^A \equiv p(t,r)\xi_0^A(x^B)$$

- If the two commute: isometry ξ_1 **lifts**

$$[\![\xi_0, \xi_1]\!]^A = \frac{1}{p(t,r)}[\![\nu, \xi_1]\!]^A = 0$$

In what follows we assume these **do not commute**!

Basic idea

- To construct irreducible KTs we assume: **noncommutativity** of KVs on S and the existence of a "**Casimir**":

$$[\![\xi_i, \xi_j]\!] = f_{ijk}\xi_k \,, \quad \text{where } i, j, k, \in \{0, 1, 2\}$$

$$[\![C_{12}, \xi_0]\!]_{\mathrm{SN}} = 0 \qquad C_{12} \equiv \xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2$$

- Then, KVs ξ_k do not lift but $\hat{C}_{12}$ is an (irreducible) **Killing tensor** on M.

Back to Lense-Thirring example: SO(d-1)

$$\nu^A = \sum_{i=1}^{m} p_i(t,r)(\partial_{\phi_i})^A$$

$$\gamma_{AB} dx^A dx^B = r^2 \left(\sum_{i=1}^{m+\epsilon} d\mu_i^2 + \sum_{i=1}^{m} \mu_i^2 d\phi_i^2 \right)$$

- So we have

$$C_I = \sum_{p \in I \subseteq S} \xi_p \otimes \xi_p$$

$$[\![C_I, \partial_{\phi_i}]\!]_{\mathrm{SN}} = 0 \quad \forall i \in \{1, \dots, m\}$$

$$[\![C_I, C_{I'}]\!]_{\mathrm{SN}}^{ABC} = 4 \sum_{\substack{p \in I, q \in I', \\ r \in S}} f_{pqr}\, \xi_p^{(A} \xi_q^{B} \xi_r^{C)}$$

$$\left[\!\!\left[[\![\cdots [\![C_{I_1}, C_{I_2}]\!], \cdots]\!], C_{I_k} \right]\!\!\right] = -(-2)^k \prod_{i=0}^{k-2} \sum_{\substack{s_j \in I_j \\ r_j \in S \\ t_i \in \{s_1,\dots,s_{i+1},r_1,\dots,r_i\} \setminus \\ \{t_1,\dots,t_{i-1}\}}} f_{s_{i+2} t_i r_{i+1}} \bigodot_{\substack{u \in \{s_1,\dots,s_k, \\ r_1,\dots,r_{k-1}\} \setminus \\ \{t_1,\dots,t_{k-2}\}}} \xi_u$$

- The tower of increasing rank tensors **does not** obviously **terminate**. **Irreducibility** comes from **non-commutativity** of KVs. (Generate finite number of functionally independent integrals of motion.)

Q3: On shell realization

- Rotating black hole in EMDA theory (Clement, Galtsov, Leygnac 2002)

$$g = -N\tilde{f}dt^2 + \frac{dr^2}{\tilde{f}} + r^2 \sin^2\theta (d\varphi - \frac{a}{r^2}dt)^2 + r^2 d\theta^2$$

$$\tilde{f} = \frac{1}{4} - \frac{mr_0}{2r^2} + \frac{a^2 r_0^2}{4r^4}, \quad N = \frac{4r^2}{r_0^2},$$

Q4: Other examples easy to construct

- Base with planar symmetry
- Base identified with (Euclidean) Taub-NUT spacetime

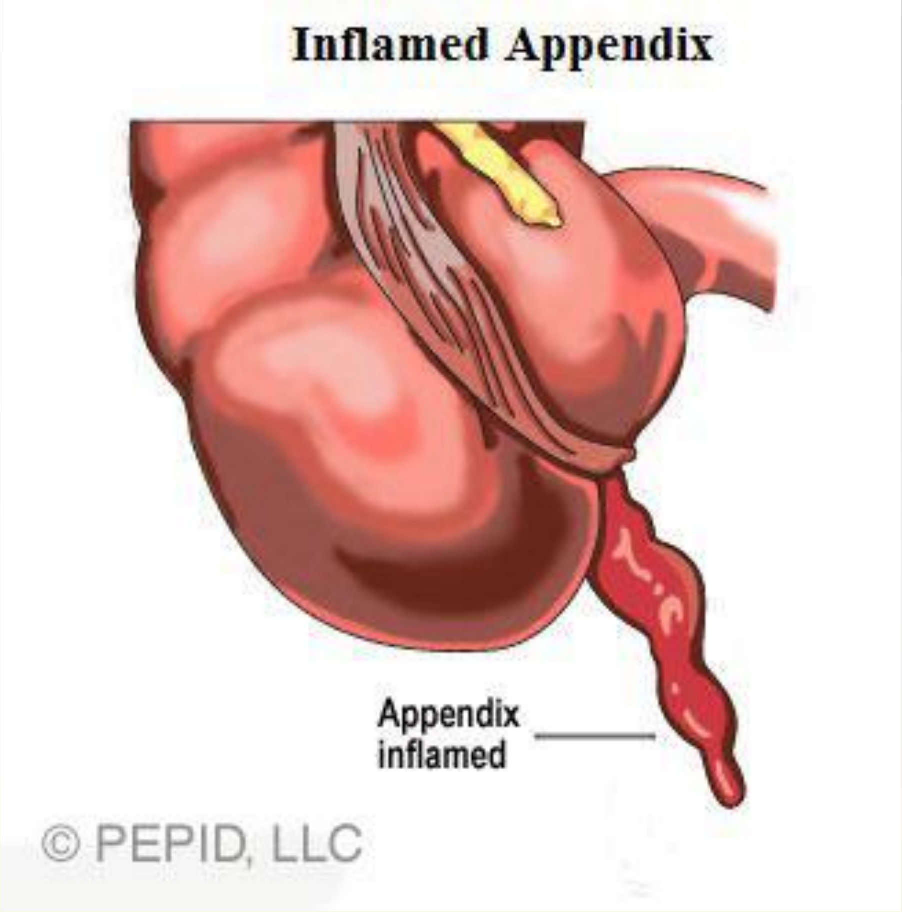
Summary

1) **Dynamical symmetries** are genuine phase space symmetries that play interesting role in many areas of physics. They are **hidden** in configuration space. In GR these are described by **Killing** tensors (and also **Killing-Yano** tensors).

2) "**Magic square**" version of Lense-Thirring spacetimes admits a **remarkable tower** of rank-2 and **higher-rank** (irreducible) Killing tensors – first example of **physically motivated** spacetime with higher-rank KTs.

3) Such non-trivial Killing tensors arise from products of **non-commuting** Killing vectors of the base space. The tower does not seem to terminate. **Other examples** can be easily constructed.

Appendices



Painleve-Gullstrand (PG) form

$$ds^2 = - \quad f dt^2 + \frac{dr^2}{f} + r^2 \sin^2\theta \left(d\phi + \frac{ap}{r^2} dt \right)^2 + r^2 d\theta^2$$

- Radially infalling observer (from rest at "infinity")

$$u = dt + \frac{\sqrt{1-f}}{f} dr$$

$$dt = dT - \frac{\sqrt{1-f}}{f} dr$$

$$ds^2 = -dT^2 + (dr + \sqrt{1-f}\, dT)^2 + r^2 d\theta^2 + r^2 \sin^2\theta \left(d\phi + \frac{a(f-1)}{r^2} dT - \frac{a(f-1)\sqrt{1-f}}{r^2 f} dr \right)^2$$

- Finally set $d\phi = d\Phi + \frac{a(f-1)\sqrt{1-f}}{r^2 f} dr$

$$ds^2 = -dT^2 + (dr + \sqrt{1-f}\, dT)^2 + r^2 d\theta^2 + r^2 \sin^2\theta \left(d\Phi + \frac{a(f-1)}{r^2} dT \right)^2 ,$$

K. Martel, E. Poisson, gr-qc/0001069.

Upgraded LT spacetimes

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\phi_i + \frac{a_i p_i}{r^2} dt \Big)^2 + r^2 (\sum_{i=1}^{m} d\mu_i^2) + \epsilon r^2 d\nu^2$$

- Horizon

$$f(r_+) = 0$$

$$\xi = \partial_t + \sum_{i=1}^{m} \Omega_i \partial_{\phi_i}\,, \qquad \Omega_i = -\frac{a_i p_i}{r^2}\Big|_{r=r_+}$$

(ergosphere)

- PG form

$$ds^2 = -N dT^2 + \Big(dr + \sqrt{N(1-f)} dT \Big)^2 + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\Phi_i + \frac{a_i p_i}{r^2} dT \Big)^2 + r^2 (\sum_{i=1}^{m} d\mu_i^2) + \epsilon r^2 d\nu^2\,,$$

(horizon is manifestly regular)

- Explicit symmetries ∂_t ∂_{ϕ_i} (m+1)… can be enhanced

Towers of hidden symmetries

- Explicitly

$$K^{(I)} = \sum_{i \notin I}^{m-1+\epsilon} \left[\left(1 - \mu_i^2 - \sum_{j \in I} \mu_j^2\right) (\partial_{\mu_i})^2 - 2 \sum_{j \notin I \cup \{i\}} \mu_i \mu_j \, \partial_{\mu_i} \partial_{\mu_j} \right] + \sum_{i \notin I}^{m} \left[\frac{1 - \sum_{j \in I} \mu_j^2}{\mu_i^2} (\partial_{\phi_i})^2 \right] . \qquad (30)$$

- Of these

$$k = \sum_{i=0}^{m-2+\epsilon} \binom{m}{i} - \sum_{i=0}^{m-3} \binom{m}{i} = \frac{1}{2} m(m-1+2\epsilon)$$

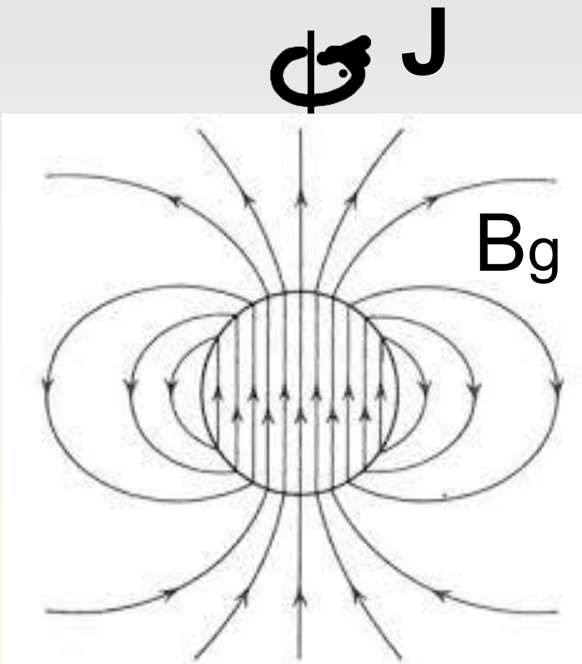
… are **irreducible**

- Note that this **tower grows quadratically** with number of dimensions (compared to linear for Kerr-AdS)
- For high enough d, provides an example of a spacetime with **more hidden** than explicit symmetries

Frame dragging (gravitomagnetism)

= general-relativistic effect due to the **motion** (in particular rotation) **of matter** and gravitational waves, analogous in a way to electromagnetic induction.

- **Lens-Thirring (1918)**



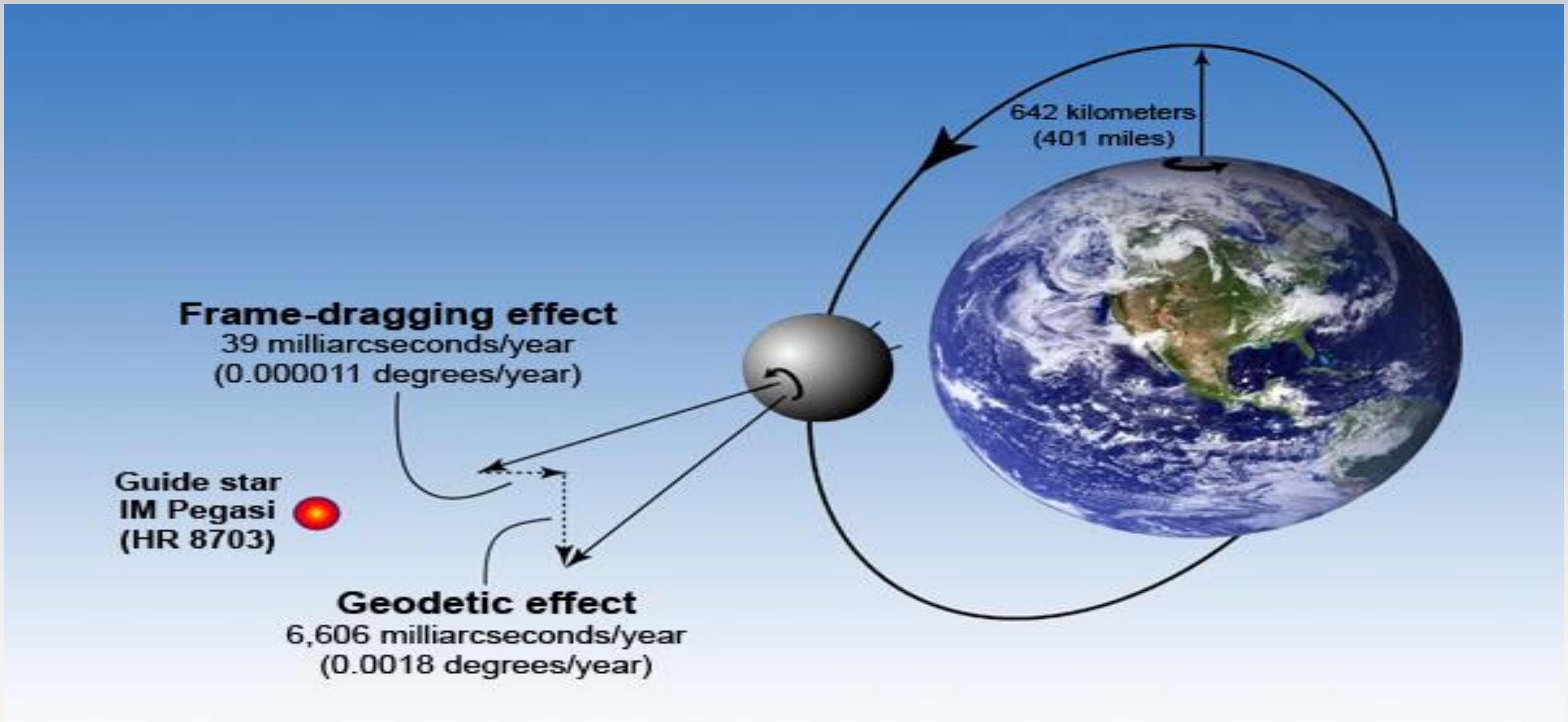
- "radially infalling geodesic" experiences "**Coriolis type force**"

$$r\frac{d^2\varphi}{dt^2} = -\underbrace{\frac{2J}{r^3}}_{2\omega(r)}\frac{dr}{dt}$$

- **Gyroscope precession** ("Larmor precession" due to gravitomagnetic field)

The Gravity Probe B Experiment

Everitt; et al. *"Gravity Probe B: Final Results of a Space Experiment to Test General Relativity".* Phys. Rev. Lett. **106** (22): 221101 (2011)



	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2

https://physics.aps.org/articles/v4/43