Topos-theoretic extension of vacuum algebraic quantum field theory over curved space-times

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dedicated to the memory of

Marek Zawadowski (1960–2024)

Jurek Lewandowski (1959–2024)

I. Outline

1) Causal logics:
   subsets of a lorentzian space-time, closed w.r.t. some form of causal signalling (e.g., by time-like curves), form orthocomplemented lattices.
2) Haag's «tentative postulate»:
   vacuum sector of a.q.f.t. over Minkowski space-time given by a homomorphism from a causal logic into a lattice of factor von Neumann subalgebras.
3) Weakening of Haag's postulate:
   not a homomorphism, but a Galois connection (a pair of adjoint functors between lattices: one preserves $\vee$, another preserves $\wedge$).
4) Spectral presheaf:
   a presheaf of Stone spectra of boolean subalgebras of the orthocomplemented lattice, and embedding of lattice into this presheaf.
5) Spectral presheaf extension of causal logics and von Neumann subfactor lattices:
   a) paraconsistency of nonsignalling
   b) causal boundary operator
   c) emergence of causal sublattice
   d) nontrivial bi-Heyting modal operators $\leftrightarrow$ closed time-like curves (?)

II. Main motivations

1) Spectral presheaf formalism was so far applied only to the lattices of projections in von Neumann algebras, in the context of foundations of quantum mechanics. Our work brings it into new contexts of causal structure of lorentzian space-times, and vacuum a.q.f.t. over them.
2) The lattice structure of causal logics encodes only some aspects of the causal structure of space-time. Extension with spectral presheaves should allow to encode more structure.
3) Haag's «tentative postulate» is generally too strong. The proposed weakening is natural from the category-theoretic perspective, and combines well with the spectral presheaf formalism. So it may lead to some new insights.

III. Orthocomplemented lattices

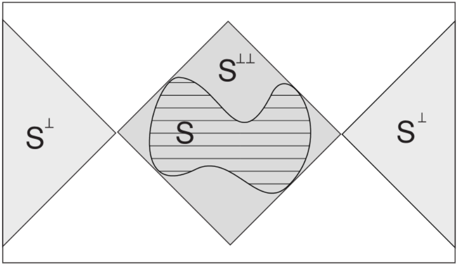
von Neumann'32, Birkhoff–von Neumann'36, Husimi'37, Maeda'55, Loomis'55

- A partially ordered set $(L, \leq)$ is a bounded lattice iff
  - $\exists$ a supremum/join $x \vee y \in L$ $\forall x, y \in L$,
  - $\exists$ an infimum/meet $x \wedge y \in L$ $\forall x, y \in L$,
  - $\exists$ a greatest element $1 \in L$,
  - $\exists$ a smallest element $0 \in L$.
- A bounded lattice is complete iff all of its subsets have meets and joins.
- A bounded lattice is orthocomplemented iff $\forall x \in L$ $\exists x^{\perp} \in L$ s.t.
  - $x^{\perp\perp} = x$,
  - $x^{\perp} \vee x = 1$, (equivalently: $x^{\perp} \wedge x = 0$)
  - if $x \leq y$ then $x^{\perp} \geq y^{\perp}$ $\forall y \in L$.
- An orthocomplemented lattice $(L, ^{\perp})$ is orthomodular iff $x \vee y = ((x \vee y) \wedge y^{\perp}) \vee y$ $\forall x, y \in L$.
- A boolean algebra is an orthomodular lattice $(L, ^{\perp})$ satisfying $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $\forall x, y, z \in L$.
- Example: The set of all projections on a Hilbert space, or in any W*-algebra, is an orthomodular lattice: $0 := 0$, $1 := \mathbb{I}$, $P \leq Q \;:=\; P = PQ$, $P^{\perp} := \mathbb{I} - P$, $\mathrm{ran}(P \vee Q) := \mathrm{ran}(P) \cup \mathrm{ran}(Q)$, $P \wedge Q := (P^{\perp} \vee Q^{\perp})^{\perp}$. Its boolean subalgebras are the same as the sets of mutually commuting projectors.

IV. Causal logics (I)

Cegła–Jadczyk'77'79, Cegła–Florek'79'81'05'06, Casini'02'03, Cegła–Florek–Jancewicz'17

- Let (M, g) := arbitrary (≥ 2)-dimensional lorentzian space-time.
- For $S \subseteq M$, let $S^{\perp} :=$ {all $x \in M$ not connected with S by a time-like curve}.



source: H. Casini, 2002, Class. Quant. Grav. **19**, 6389-6404.

- The set $L_{(M,g)}$ of subsets $S \subseteq M$, s.t. $S = S^{\perp\perp}$, equipped with
  - $S_1 \leq S_2 := S_1 \subseteq S_2$,
  - $S_1 \wedge S_2 := S_1 \cap S_2$,
  - $S_1 \vee S_2 := (S_1 \cup S_2)^{\perp\perp}$,

  is a complete orthomodular lattice.
- Boolean subalgebras of $L_{(M,g)}$ = sets generated by $(\cdot)^{\perp\perp}$ from the subsets of achronal surfaces of (M, g).

V. Causal logics (II)

Cegła'89, Casini'02'03, Nobili'06

- If $S^{\perp} := \{\text{all } x \in M$ not connected with S by a time-like or null-like curve$\}$, then $L_{(M,g)}$ is orthocomplemented but not orthomodular.
- Lattices defined by discretised space-times are also orthocomplemented but not orthomodular.
- Nobili'06: «(...) it is difficult to define unambiguously (...) the boundaries of causal completions».
- $L_{(M,g)}$ does not satisfy so-called "covering property"[1], that is always satisfied by the orthomodular lattice of projections $L_{\mathrm{Proj}(\mathcal{N})}$ in any W*-algebra $\mathcal{N}$, so $L_{(M,g)}$ cannot be represented by $L_{\mathrm{Proj}(\mathcal{N})}$.

[1] An orthomodular lattice $(L, ^{\perp})$ has the covering property iff

$$\forall x, y \in L \ \text{ if } (y \text{ is an atom}, y \vee y^{\perp} \neq 1) \text{ then } x \wedge (x^{\perp} \vee y) \text{ is an atom},$$

where $z \in L$ is an atom iff $0 < z$ and there exists no $w \in L$ s.t. $0 < w < z$.

VI. Orthocomplemented lattice of factor sub-W*-algebras

von Neumann'29, Haag–Schroer'62, Haag–Kastler'64, ..., Haag'92/'96

- Given a W*-algebra $\mathcal{N}$, let $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ be sub-W*-algebras of $\mathcal{N}$.
- A commutant of $\mathcal{A}$ in $\mathcal{N}$ $:= \mathcal{A}^\bullet := \{x \in \mathcal{N} : xy = yx \; \forall y \in \mathcal{A}\}$.
- $\mathcal{A}_1 \wedge \mathcal{A}_2 :=$ the largest sub-W*-algebra of $\mathcal{N}$ contained in $\mathcal{A}_1$ and $\mathcal{A}_2$.
- $\mathcal{A}_1 \vee \mathcal{A}_2 :=$ the smallest sub-W*-algebra of $\mathcal{N}$ containing $\mathcal{A}_1$ and $\mathcal{A}_2$.
- This implies:
  - $\mathcal{A}_1 \wedge \mathcal{A}_2 = \mathcal{A}_1 \cap \mathcal{A}_2$; $\quad \mathcal{A}_1 \vee \mathcal{A}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2)^{\bullet\bullet}$;
  - $(\mathcal{A}_1 \wedge \mathcal{A}_2)^\bullet = \mathcal{A}_1^\bullet \vee \mathcal{A}_2^\bullet$; $\quad (\mathcal{A}_1 \vee \mathcal{A}_2)^\bullet = \mathcal{A}_1^\bullet \wedge \mathcal{A}_2^\bullet$;
  - $\mathcal{N}^\bullet = \mathbb{C}\mathbb{I}$; $\quad (\mathbb{C}\mathbb{I})^\bullet = \mathcal{N}$.
- $\mathcal{A}$ is a factor iff $\mathcal{A} \cap \mathcal{A}^\bullet = \mathbb{C}\mathbb{I}$ ($\iff \mathcal{A} \vee \mathcal{A}^\bullet = \mathcal{N}$).
- Hence: the set $L_{\mathcal{N}}$ of factor sub-W*-algebras of a factor W*-algebra $\mathcal{N}$, equipped with $(\vee, \wedge, ^\bullet, 0 := \mathbb{C}\mathbb{I}, 1 := \mathcal{N})$ is an orthocomplemented lattice.

VII. Vacuum algebraic q.f.t.: Haag's «tentative postulate»

Araki'61, Haag–Schroer'62, Haag–Kastler'64, ..., Haag'92/'96

- Minimal setting for algebraic q.f.t.:
  1) a functor $\mathfrak{A}$ from the category of subsets of space-time with embeddings as morphisms to the category of sub-W*-algebras of a W*-algebra with embeddings as morphisms,
  2) if $S_1 \subseteq S_2$ then $\mathfrak{A}(S_1) \subseteq \mathfrak{A}(S_2)$,
  3) if $S = \bigcup_j S_j$ then $\mathfrak{A}(S) = \bigvee_j \mathfrak{A}(S_j) = (\bigcup_j \mathfrak{A}(S_j))^{\bullet\bullet}$,
  4) if $S_1 \subseteq S_2^{\perp}$ then $\mathfrak{A}(S_1) \subseteq (\mathfrak{A}(S_2))^{\bullet}$ (=: causality).
- Haag–Schroer duality property := $(\mathfrak{A}(S^{\perp}) = (\mathfrak{A}(S))^{\bullet})$.
- Haag'92/'96 «tentative postulate»:
  1) consider an orthomodular lattice $(L_{(M,g)}, {}^{\perp})$, where (M, g) is a Minkowski space-time, and ${}^{\perp}$ is a time-like nonsignalling;
  2) consider an orthocomplemented lattice $(L_{\mathcal{N}}, {}^{\bullet})$ of factor sub-W*-algebras of a factor W*-algebra $\mathcal{N}$;
  3) an algebraic q.f.t., for the vacuum sector of the theory, is given by an orthocomplemented lattice homomorphism $(L_{(M,g)}, {}^{\perp}) \to (L_{\mathcal{N}}, {}^{\bullet})$.
- In general, Haag's «tentative postulate» is too strong:
  1) the Haag–Schroer duality does not hold in several models;
  2) $\wedge$-preservation is usually not required and not verified.

VIII. Vacuum pre-a.q.f.t.: beyond Haag's «tentative postulate»

RPK'24

- Let (M, g) be any lorentzian space-time, and $(L_{(M,g)}, {}^{\perp})$ be a causal logic.
- Consider the following categorical reformulation, and weakening, of Haag's postulate:

  a vacuum pre-a.q.f.t. := an injective, $(0, 1, \vee)$-preserving $\leq$-monotone functor $\mathfrak{N}^{\flat} : (L_{(M,g)}, {}^{\perp}) \to (L_{\mathcal{N}}, {}^{\bullet})$, satisfying $\mathfrak{N}^{\flat}((\cdot)^{\perp}) \leq (\mathfrak{N}^{\flat}(\cdot))^{\bullet}$ ("causality").

- By the adjoint functor theorem, $\mathfrak{N}^{\flat}$ has a surjective, $(0, 1, \wedge)$-preserving $\leq$-monotone adjoint $\mathfrak{N}^{\sharp} : (L_{\mathcal{N}}, {}^{\bullet}) \to (L_{(M,g)}, {}^{\perp})$, i.e.

$$\mathfrak{N}^{\flat}(x) \leq y \iff x \leq \mathfrak{N}^{\sharp}(y),$$

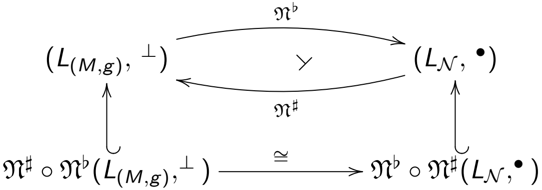
  so $\mathfrak{N}^{\flat} \dashv \mathfrak{N}^{\sharp}$ is a monotone Galois connection.

- We will say that $\mathfrak{N}^{\flat}$ satisfies the Haag–Schroer duality iff $\mathfrak{N}^{\flat}((\cdot)^{\perp}) = (\mathfrak{N}^{\flat}(\cdot))^{\bullet}$.

IX. Emergence in vacuum pre-a.q.f.t. (I)

RPK'25

- Haag's «tentative postulate» of an orthocomplemented lattice homomorphism $H : (L_{(M,g)}, ^{\perp}) \to (L_{\mathcal{N}}, ^{\bullet})$ contains implicitly a statement of emergence of the causal logic structure from the structure of a factor sub-W*-algebras for a sublattice of $(L_{(M,g)}, ^{\perp})$ on which H is an isomorphism.
- The vacuum pre-a.q.f.t. $\mathfrak{N}^{\flat} : (L_{(M,g)}, ^{\perp}) \to (L_{\mathcal{N}}, ^{\bullet})$, when combined with its right adjoint functor $\mathfrak{N}^{\sharp} : (L_{\mathcal{N}}, ^{\bullet}) \to (L_{(M,g)}, ^{\perp})$, induces an equivalence of sublattices defined by $\{x \in L_{(M,g)} : \mathfrak{N}^{\sharp} \circ \mathfrak{N}^{\flat}(x) \leq x\}$ and $\{y \in L_{\mathcal{N}} : y \leq \mathfrak{N}^{\flat} \circ \mathfrak{N}^{\sharp}(y)\}$, which can be seen as emergence over a restricted domain:



- We will call it a strong emergence, if $\mathfrak{N}^{\flat}$ satisfies the Haag–Schroer duality.

X. Spectral presheaf

Isham–Butterfield'98, Döring–Isham'08, Cannon'13, Cannon–Döring'18

- Stone'36 duality:
  - Every boolean algebra B determines a Stone space $S_B :=$ a set of nonzero boolean homomorphisms $B \to \{0,1\}$, equipped with a suitable (totally disconnected compact Hausdorff) topology.
  - Given any t.d.c.H. topological space S, the set of all closed-and-open subsets of S forms a boolean algebra B_S.
  - $S_{B_S} = S$, $B_{S_B} = B$.
- Let L be a complete orthomodular lattice.
- Let $\mathbf{B}(L) :=$ a category with: $\{$objects $:=$ boolean subalgebras of L; morphisms $:=$ inclusions$\}$.
- A spectral presheaf $:=$ a contravariant functor $\Sigma_L : \mathbf{B}(L) \to \mathbf{Set}$, s.t. $\{B \mapsto S_B; (B_1 \hookrightarrow B_2) \mapsto$ restriction: $(S_{B_2} \to S_{B_1})\}$.
- $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L) :=$ set of subfunctors F of Σ_L, s.t. $F(B)$ is a closed-and-open set $\forall B \in \mathrm{Ob}(\mathbf{B}(L))$.
- $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$ is a complete lattice, when equipped with:
  $x \leq y :\iff x_B \subseteq y_B \; \forall B \in \mathrm{Ob}(\mathbf{B}(L))$,
  $(p \wedge q)_B := \mathrm{int}(p_B \cap q_B)$,
  $(p \vee q)_B := \mathrm{cl}(p_B \cup q_B)$.

XI. Outer daseinisation and paraconsistency

de Groote'05, Döring–Isham'08, Döring'16, Cannon'13, Cannon–Döring'18, Eva'15'16, Döring–Eva–Ozawa'21

- Consider: $$\delta_B(x) := \underbrace{\left\{ s \in S_B \,:\, s\left(\underbrace{\bigwedge\{y \in B \,:\, y \geq x\}}_{\text{best approx. of } x \text{ in } B}\right) = 1 \right\}}_{\text{elements of } S_B \text{ for which the best approx. of } x \text{ holds}} \;\forall B \in \mathrm{Ob}(\mathbf{B}(L)).$$
- An outer daseinisation of $x \in L$:= a contravariant functor $\delta(x) : \mathbf{B}(L) \to \mathbf{Set}$, s.t. $\{B \mapsto \delta_B(x) \subseteq S_B;\, (B_1 \hookrightarrow B_2) \mapsto$ restriction: $(S_{B_2} \to S_{B_1})\}$.
- $\delta : L \to \mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$ is an injective, $(0,1,\vee)$-preserving map.
- By an adjoint functor theorem for posets, there exists a surjective, $(0,1,\wedge)$-preserving map $\varepsilon : \mathrm{Sub}_{\mathrm{clop}}(\Sigma_L) \to L$, s.t. $\delta(x) \leq y \iff x \leq \varepsilon(y)$, i.e. $\delta \dashv \varepsilon$ is a Galois connection.
- $\multimap(\cdot) := \delta((\varepsilon(\cdot))^\perp)$ is a proper paraconsistent negation on $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$, i.e. $x \wedge \multimap x \geq 0$ and ($x \wedge \multimap x = 0$ iff $x \in \{0,1\}$). I.e. $\multimap$ does not satisfy the law of noncontradiction (*ex falso quodlibet*).
- Eva'15'16 (claim, no proof): $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \multimap)$, equipped with implication $x \Rightarrow y := \multimap x \vee y$, satisfies the axioms and rules of inference of the relevant paraconsistent logic **DL** (of Routley'77).

XII. DK and DL logics

Routley–Meyer'76, Routley'77

- Axioms of **DK**:
  - A_1) $a \triangleright a$ (*identity*),
  - A_{2a}) $a \wedge b \triangleright a$ (*left conjuctive implication*),
  - A_{2b}) $a \wedge b \triangleright b$ (*right conjuctive implication*),
  - A_3) $(a \wedge (b \vee c)) \triangleright ((a \wedge b) \vee (a \wedge c))$ (*distribution*),
  - A_4) $((a \triangleright b) \wedge (b \triangleright c)) \triangleright (a \triangleright c)$ (*conjuctive syllogism*),
  - A_5) $((a \triangleright b) \wedge (a \triangleright c)) \triangleright (a \triangleright (b \wedge c))$ (*composition*),
  - A_6) $(a \triangleright \sim b) \triangleright (b \triangleright \sim a)$ (*contraposition*),
  - A_7) $\sim\sim a \triangleright a$ (*double negation elimination*).
- Deductive inference rules of **DK**:
  - R_1) $(a,\ b) \vdash a \wedge b$ (*adjunction*),
  - R_2) $(a, a \triangleright b) \vdash b$ (*modus ponens*),
  - R_3) $((a \triangleright b), (c \triangleright d)) \vdash (b \triangleright c) \triangleright (a \triangleright d)$ (*affixing*).
- **DL** := **DK** with an additional axiom: $(a \triangleright \sim a) \triangleright \sim a$ (*reduction*).
- There are available additional axioms and rules for extending these logics with quantifiers and propositional constants.
- For further purposes we will define:
  - $\mathbf{DL}_0$:= **DL** minus A_4 minus R_2 minus R_3,
  - $\mathbf{DK}_0$:= **DK** minus A_4 minus R_2 minus R_3.

XIII. Paraconsistent logic in a spectral presheaf

RPK'24, *B.Engel–RPK'25

1) Assume: $\sim = \multimap$,
$\triangleright = \Rightarrow$, where $x \Rightarrow y := \multimap x \vee y$,
$(a, b \vdash c) = (a \wedge b \leq c)$.
Then the axiom A_4 (i.e. $((a \Rightarrow b) \wedge (b \Rightarrow c)) \Rightarrow (a \Rightarrow c))$* and two rules of inference of **DL** (affixing and *modus ponens*) are, contrary to Eva's claim, not provable in $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \multimap, \Rightarrow)$, so the latter is a model of a weaker paraconsistent logic, $\mathbf{DL}_0$.

2*) We prove a no-go theorem for *modus ponens* $((a, a \triangleright b) \vdash b)$ on $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \multimap, \triangleright)$ for a rich family of possible implications $\triangleright$ that are definable using $\multimap$ (including $\Rightarrow$).

3) Lack of *modus ponens* is problematic, since it is a main rule of deductive inference in many logical systems.

4*) We introduce a sublattice $\bigwedge \delta(L) := \{\bigwedge_n \delta(a_n) : n \in \mathbb{N}, a_n \in L\}$ of $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$, for which we establish *modus ponens*, under three different implications constructed from $\multimap$ (but not for $\Rightarrow$):
- $a \triangleright_s b := \multimap a \vee \multimap\multimap(a \wedge b)$;
- $a \triangleright_c b := \multimap b \triangleright_s \multimap a$;
- $a \triangleright_r b := \multimap\multimap((a \triangleright_s b) \wedge (a \triangleright_c b))$.

XIV. Representation of causal logic in spectral presheaf

RPK'19/'24/'25

- We postulate to study $L_{(M,g)}$ in terms of $\Sigma_{L_{(\boldsymbol{M},\boldsymbol{g})}}$ and $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(\boldsymbol{M},\boldsymbol{g})}})$.
- The Cannon–Döring–Eva construction works as well without assuming orthomodularity of L.
- The only thing that gets lost with such a weakening is characterisation of L by Σ_L up to an isomorphism.

Results:

1) Causal nonsignalling (in different variants), encoded by $^{\perp}$ on $L_{(M,g)}$, is represented, via a Galois connection $\delta \dashv \varepsilon$, by a proper paraconsistent negation $\multimap$ on $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(\boldsymbol{M},\boldsymbol{g})}})$, satisfying the rules and axioms of $\mathbf{DL}_0$ logic.
2) Since $\delta \dashv \varepsilon$ is monotone, the strengthening (resp., weakening) of nonsignalling corresponds to strengthening (resp., weakening) of paraconsistent negation.
3) We introduce a boundary operator for $\multimap$, $\partial S := S \wedge \multimap S$. It satisfies:

$$\partial(x \wedge y) = (\partial x \wedge y) \vee (x \wedge \partial y) \quad \text{(Leibniz rule)},$$
$$x = \partial x \vee \multimap\multimap x.$$

and it encodes the properties of the causal boundary of the presheaves representing causally complete sets.

XV. Spectral presheaf vacuum pre-a.q.f.t (*Jenseits von Haag und Dasein*)

RPK'24

- Combining earlier constructions, we define a spectral presheaf vacuum pre-a.q.f.t. as an injective, $(0, 1, \vee)$-preserving functor

$$\mathfrak{M}^\flat : (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(\boldsymbol{M},\boldsymbol{g})}}), \multimap) \to (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{\mathcal{N}}}), \mathrel{-\!\!\bullet}),$$

such that:
1) $\mathfrak{M}^\flat(\multimap(\cdot)) \leq \mathrel{-\!\!\bullet}(\mathfrak{M}^\flat(\cdot))$ ("paraconsistent causality"),
2) the following diagram commutes:

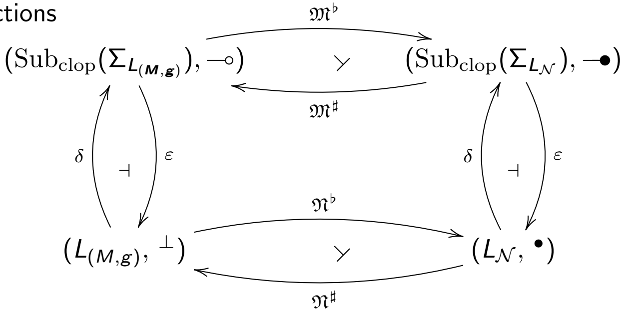
$$\begin{array}{ccc}
(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(\boldsymbol{M},\boldsymbol{g})}}), \multimap) & \xrightarrow{\mathfrak{M}^\flat} & (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{\mathcal{N}}}), \mathrel{-\!\!\bullet}) \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
(L_{(M,g)}, {}^{\perp}) & \xrightarrow{\mathfrak{N}^\flat} & (L_{\mathcal{N}}, {}^{\bullet}).
\end{array}$$

- We will say that a spectral presheaf vacuum pre-a.q.f.t. $\mathfrak{M}^\flat$ satisfies the paraconsistent Haag–Schroer duality iff $\mathfrak{M}^\flat(\multimap(\cdot)) = \mathrel{-\!\!\bullet}(\mathfrak{M}^\flat(\cdot))$.
- By the adjoint functor theorem, $\mathfrak{M}^\flat$ determines the right adjoint functor $\mathfrak{M}^\sharp : (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{\mathcal{N}}}), \mathrel{-\!\!\bullet}) \to (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(\boldsymbol{M},\boldsymbol{g})}}), \multimap)$.

XVI. Emergence in vacuum pre-a.q.f.t. (II)

RPK'25

- The adjunctions



induce the "emergent" equivalence:

$$\{x \in L_{(M,g)} : \varepsilon \circ \mathfrak{M}^{\sharp} \circ \mathfrak{M}^{\flat} \circ \delta(x) \leq x\} \cong \{y \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{\mathcal{N}}}) : y \leq \mathfrak{M}^{\flat} \circ \delta \circ \varepsilon \circ \mathfrak{M}^{\sharp}(y)\},$$

which is "strong" iff $\mathfrak{M}^{\flat}$ satisfies the paraconsistent Haag–Schroer duality.

- If $\mathfrak{M}^{\sharp}$ satisfies the inverse of paraconsistent Haag–Schroer duality, i.e. $\mathfrak{M}^{\sharp}(-\bullet(\cdot)) = \multimap(\mathfrak{M}^{\sharp}(\cdot))$, then the subfactor boundaries $\overset{\bullet}{\partial}\mathcal{A}$ are mapped by $\mathfrak{M}^{\sharp}$ into the causal boundaries $\overset{\circ}{\partial}(\mathfrak{M}^{\sharp}(\mathcal{A}))$.

XVII. Summary

1) Clarification of the structure of intrinsic paraconsistent logic of (any) spectral presheaves
2) Spectral presheaves over causal logics:
   a) Paraconsistency of nonsignalling
   b) Causal boundary operator
3) Spectral presheaves over factor von Neumann subalgebras
4) Category-theoretic weakening of Haag's «tentative postulate» on vacuum a.q.f.t.
5) Lifting this vacuum a.q.f.t. ansatz to a pair of adjoint functors between the respective spectral presheaves
6) A prescription for emergence of causal sublogic from factor von Neumann algebra sublattice in both cases

XVIII. Bi-Heyting algebras

Skolem'1919, Zarycki'27, Heyting'30, Birkhoff'40, McKinsey–Tarski'46, Klemke'71, Rauszer'71'74, Lawvere'76,'86,'89,'91

- A bounded latice $(L, \leq, \wedge, \vee, 0, 1)$ is called:
  - Heyting iff $\forall x, y \in L\ \exists!\ x \rightarrow y \in L\ \forall z \in L$
  $$z \leq x \rightarrow y \quad :\Longleftrightarrow \quad z \wedge x \leq y;$$
  - co-Heyting iff $\forall x, y \in L\ \exists!\ x \prec y \in L\ \forall z \in L$
  $$z \geq x \prec y \quad :\Longleftrightarrow \quad z \vee y \geq x;$$
  - bi-Heyting iff it is Heyting and co-Heyting.
- Defining $\neg x := x \rightarrow 0$ and $\llcorner x := 1 \prec x$ (i.e. $\neg x :=$ the largest element of L s.t. $\neg x \wedge x = 0$; $\llcorner x :=$ the smallest element of L s.t. $\llcorner x \vee x = 1$), we get:
  - $\neg x \vee x \leq 1$, i.e. $\neg$ does not satisfy *tertium non datur* (law of excluded middle),
  - $\llcorner x \wedge x \geq 0$, i.e. $\llcorner$ does not satisfy *ex falso quodlibet* (law of noncontradiction).
- In general, logics invalidating *ex falso quodlibet* are called paraconsistent.
- Zarycki–Lawvere boundary operator, $\partial(\cdot) := (\cdot) \wedge \llcorner(\cdot)$, satisfies:

$$\partial(x \wedge y) = (\partial x \wedge y) \vee (x \wedge \partial y) \quad \text{(Leibniz rule)},$$
$$x = \partial x \vee \llcorner\llcorner x.$$

XIX. Nontrivial modality implies closed time-like (or vertex) curves?

RPK'19/'24, *B.Engel–RPK'25

Modal operators in $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(M,g)}})$ vs closed time-like/vertex curves in (M, g):

a) $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$ has a structure of a bi-Heyting algebra [Döring'11].

b) Every bi-Heyting algebra H allows to construct modal operators $\Box : H \to H$ and $\Diamond : H \to H$, defined by $\Box := \bigwedge_{n\in\mathbb{N}} \Box_n$ and $\Diamond := \bigvee_{n\in\mathbb{N}} \Diamond_n$, where $\Box_0 := \mathrm{id}_H =: \Diamond_0$, $\Box_{n+1} := \neg\lrcorner\Box_n$, and $\Diamond_{n+1} := \lrcorner\neg\Diamond_n$ [Reyes–Zolfaghari'96].

c) For a W*-algebra $\mathcal{N}$ and L given by a lattice of projections of $\mathcal{N}$, $\Box$ and $\Diamond$ in $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \neg, \lrcorner)$ are nontrivial only when $\mathcal{N}$ has a nontrivial center [Eva'16].

d) For the time-like nonsignalling orthomodular variant of $\perp$, $L_{(M,g)}$ has nontrivial center iff (M, g) contains closed time-like (or vertex) curves (i.e. iff $\exists q \in M$ such that $I^+(q) \cap I^-(q) \neq \varnothing$) [Casini'02].

Combining the above points, we arrive to:

1) Conclusion: if an analogue of c) for orthomodular lattices holds, then nontrivial bi-Heyting modal operators in $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(M,g)}})$ would imply the presence of closed time-like (or vertex) curves in (M, g).

2*) However: it turns out that Eva's claim c) is almost certainly wrong, and so the above reasoning is, by now, nonconclusive.

3*) The benefit: we are working on describing the general structure of the bi-Heyting modal operators in any spectral presheaves. To be continued...