Jurek's Uniqueness

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Reminiscences: Six Years Ago – So True!

Jurek has found, created, inspired strong **unique**ness results.

Theorem: Jurek is **unique**.

Proof: Obvious. **qed**

- Given: classical system with first-class constraints

1. Elementary Variables
   - choose separating space $\mathfrak{S}$ of phase space functions
2. Quantization
   - choose "representation" of $\mathfrak{S}$ on some kinematical Hilbert space $\mathcal{H}$, giving self-adjoint constraints
3. Group Averaging
   - choose constraint-invariant dense subset Φ in Hilbert space $\mathcal{H}$
   - solve constraints using Gelfand triple $\Phi \subseteq \mathcal{H} \subseteq \Phi'$

$$\eta(\phi) \;\; := \;\; \int_{\mathcal{Z}} \mathrm{d}\mu(Z) \; \overline{Z\phi} \;\; \in \;\; \Phi'$$

4. Physical Hilbert Space
   - inner product: $\langle \eta\phi_1, \eta\phi_2 \rangle_{\text{phys}} \;\; := \;\; (\eta\phi_1)[\phi_2]$
   - completion of $\eta(\Phi)$ gives physical Hilbert space, self-adjoint dual representation of observable algebra

Ashtekar, Lewandowski, Marolf, Mourão, Thiemann 1995

- Choices

$\mathfrak{F}$ ... some separating set of bounded functions on $\mathcal{C}$ $\quad f(q)$

$\mathfrak{A} := \langle \mathfrak{F} \rangle$ and $\mathfrak{B} := \{a \in \mathfrak{A} \mid \mathfrak{X}^n a \subseteq \mathfrak{A} \; \forall n\}$

$\mathfrak{X}$ ... some set of $*$-invariant derivations on $\mathfrak{B}$ $\quad \{p, \bullet\}$

$\mathfrak{I}$ ... tensor $*$-algebra in $\mathfrak{B} \cup \mathfrak{X}$ factorized by the relations

$$\begin{aligned} b \cdot X - X \cdot b &= \mathrm{i}\, X(b) \\ b_1 \cdot b_2 &= b_1\, b_2 \\ X_1 \cdot X_2 - X_2 \cdot X_1 &= [X_1, X_2] \end{aligned}$$

- **Master Uniqueness Theorem**

$\mathfrak{X}(\mathfrak{F})$ and $\mathbf{1}$ span a dense subset of $\mathfrak{A}$, and ....

$\Longrightarrow$ there is **at most one** state ω on $\mathfrak{I}$ with $\omega(X^2) = 0$ for all $X \in \mathfrak{X}$

- Lemma $\qquad \Longrightarrow \; \omega|_{\mathfrak{B}}$ continuous

- Choices

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- **Master Uniqueness Theorem**

$\mathfrak{X}(\mathfrak{F})$ and $\mathbf{1}$ span a dense subset of $\mathfrak{A}$, and $\mathfrak{B}$ admits square roots

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$b \in \mathfrak{B}$ real, $||b|| < 1 \Longrightarrow \mathbf{1} - b$ has square root in $\mathbb{C}\mathbf{1} + \mathfrak{B}$

- Lemma $\qquad$ $\mathfrak{B}$ admits square roots $\Longrightarrow$ $\omega|_{\mathfrak{B}}$ continuous

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$\mathfrak{F}$ ... some separating set of bounded functions on $\mathcal{C}$ $f(q)$

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$\mathfrak{X}(\mathfrak{F})$ and $\mathbf{1}$ span a dense subset of $\mathfrak{A}$, and $\mathfrak{X}$ fulfills chain rule

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- Lemma $\mathfrak{X}$ fulfills chain rule $\Longrightarrow$ $\mathfrak{B}$ admits square roots $\Longrightarrow$ $\omega|_{\mathfrak{B}}$ continuous

$X(g \circ b) = [g' \circ b] \cdot X(b)$ for all $X \in \mathfrak{X}$, $b \in \mathfrak{B}$ and analytic g with $g(0) = 0$

- Choices

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- Prototype $\quad G$ semisimple compact Lie group, $\quad \mathfrak{F} := \{$matrix fcts to irreps$\}$
  - Peter-Weyl $\Longrightarrow$ $\mathfrak{g}(\mathfrak{F})$ and $\mathbf{1}$ span dense subset in $\mathfrak{A} \equiv \langle \mathfrak{F} \rangle = C(G)$
  - f matrix function to irrep φ with Casimir eigenvalue c_φ $\quad \curvearrowright$

either: $c_\varphi = 0 \quad \curvearrowright \quad \varphi$ trivial $\quad \curvearrowright \quad f = \mathbf{1}$

or: $c_\varphi \neq 0 \quad \curvearrowright \quad f = \frac{1}{c_\varphi} \sum_k \tau_k(\tau_k f)$ for Killing-ONB $\{\tau_k\}$ of $\mathfrak{g}$

- Choices

$\mathfrak{F}$ ... some separating set of bounded functions on $\mathcal{C}$ $f(q)$

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$\mathfrak{J}$ ... tensor $*$-algebra in $\mathfrak{B} \cup \mathfrak{X}$ factorized by the relations

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- **Master Uniqueness Theorem**

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	LQG	LQC		
		FRW ($k = 0$)		Bianchi I
config space	$\mathcal{A}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^3$
paths	analytic	analytic	linear	linear
$\mathfrak{F}$ contains	$f_\gamma \circ \pi_\gamma$	$e^{i\lambda c}$; $e^{-(c-c_0)^2}$	$e^{i\lambda c}$	$e^{i\boldsymbol{\lambda}\mathbf{c}}$
generate $\mathfrak{A}$	$C(\overline{\mathcal{A}})$	$C_{AP} + C_0$	C_{AP}	C_{AP}
$\mathfrak{X}$ contains	$X_{S,f}$	$\frac{d}{dc}$	$\frac{d}{dc}$	∂_{c_i}
diffeos	semianalytic	dilations	dilations	vol-pres dilations
uniqueness (HFA)	LOST_{05}	$\text{EHT}_{16}/\text{F}_{18}$	$\text{EHT}_{16}/\text{F}_{18}$	EHT_{17}
proof	de ns ○ 2	de ns ○ 2	de ns ○ 2	de ns ○ 2
uniqueness (Weyl)	F_{04}		$(\text{ACH})_{11}$	AC_{12}

Algebra

1. $\mathfrak{X}(\mathfrak{F})$ and $\mathbf{1}$ span dense subset of $\mathfrak{A}$
2. $\mathfrak{X}$ fulfills chain rule

Geometry

ω diffeoinv $\implies$ $\omega(X^2) = 0$

Thank You, Jurek!

Jurek has found, created, inspired strong **unique**ness results.

Theorem: Jurek is **unique**.

Proof: **Obvious.** **qed**