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GEOMETRY OF CLASSICAL AND
QUANTUM SPACE-TIMES:
JERZY LEWANDOWSKI MEMORIAL
CONFERENCE

SPIN FOAMS AND PONZANO-REGGE

My motivation in giving this talk:

1. To introduce **spin foams** – the path integral formulation of loop quantum gravity – to this audience enough for them to appreciate Jerzy's contributions.
2. To go over the issue of the **sum over local orientations** in spin foams and associated questions about **correct equations of motion in the classical limit**.
3. To look for **guidance** in the original 3D spin-foam model of quantum gravity, **Ponzano-Regge**:
   a. A similar sum over local orientations is present, also seeming to lead to incorrect equations of motion.
   b. But we know it has the correct equations of motion – flatness. Or do we?
   c. **Understand this paradox first** in order to understand the issue in the 4D case.

OUTLINE

I. **Spin foams:** Motivation

II. **Derivation of amplitude from idea of Plebanski**: Gravity as constrained topological field theory.

III. **Resulting models:** EPRL and generalizations

IV. **Contributions of Jerzy and collaborators**: To bring the covariant and canonical formulations of loop quantum gravity closer.

V. **Classical limit of EPRL**: Sum over local orientations and concern with equations of motion.

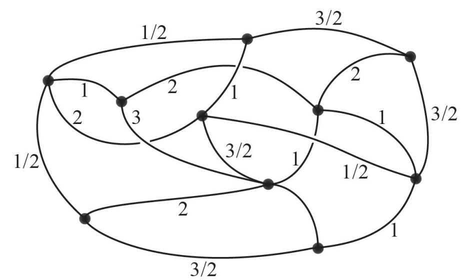
VI. **Exactly analogous phenomenon in Ponzano-Regge**, where equations of motion seem correct. **A clear paradox to learn from to guide next steps.**

I. Spin foams: Motivation

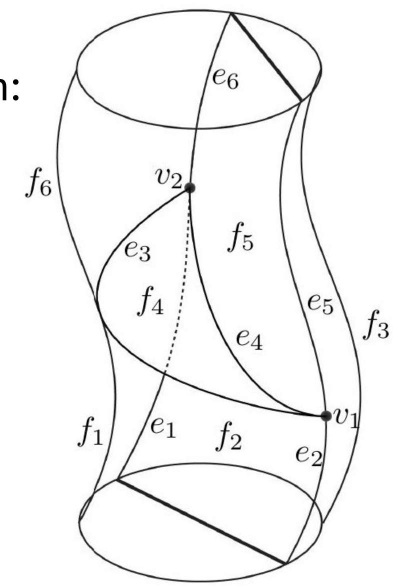
a. Path integral formulation of loop quantum gravity
b. Desire for manifestly space-time covariant formulation of dynamics (as for all path integral approaches)
c. Provides projector onto physical states: Avoid explicitly solving for the full solution to the Hamiltonian constraint operator.

- Basis of canonical loop quantum gravity: **Spin networks**
- History of elements of this basis: **Spin foam**

Spin network:



Spin foam:



II. Derivation of amplitude from idea of Plebanski: Gravity as constrained topological field theory.

Classical BF theory

$$S_{BF}[B^{IJ}_{\mu\nu}, \omega^{IJ}_{\mu}] := \int \mathrm{tr}(B \wedge F(\omega))$$

$$\delta S_{BF} := \int \mathrm{tr}\left(\delta B \wedge F(\omega) + B \wedge d_\omega \delta\omega\right) = \int \mathrm{tr}\left(\delta B \wedge F(\omega) - (d_\omega B) \wedge \delta\omega\right)$$

$$\Rightarrow \qquad F \approx 0 \qquad \text{and} \qquad d_\omega B \approx 0 \qquad \Rightarrow \qquad \text{no local degrees of freedom}$$

Simplicity with Immirzi parameter ⇒ Holst gravity

$$B^{IJ} = \frac{1}{8\pi G}\left((\star e \wedge e)^{IJ} + \frac{1}{\gamma} e^I \wedge e^J\right) \qquad \text{for some} \qquad e^I_\mu$$

$$\Rightarrow S_{BF}[B, \omega] = \frac{1}{8\pi G}\int \left((\star e \wedge e) \wedge F + \frac{1}{\gamma} e \wedge e \wedge F\right) = S_{Holst}[e, \omega]$$

II. Derivation of amplitude from idea of Plebanski: Gravity as constrained topological field theory.

Original Plebanski formulation:
- $\gamma = i$ (self-dual action), using spinorial variables
- imposed via Lagrange multiplier

Quantum BF theory: Ooguri model

Discretize on a 2-complex with 4-cells v^*, 3-cells e^* and 2-cells f^*, and $g_e \in SL(2,\mathbb{C})$:

$$\text{“}\int \mathcal{D}B\mathcal{D}\omega \exp\left(iS[B,\omega]\right) = \int \mathcal{D}B\mathcal{D}\omega \exp\left(i\int B\wedge F(\omega)\right) = \int \mathcal{D}\omega\,\delta(F(\omega)) = \text{”}$$

$$\int \left(\prod_e \mathrm{d}g_e\right)\prod_f \delta\left(\vec{\prod}_{e\in f} g_e\right) = \sum_{\{\rho_f,k_f,j_{fe}\}} \left(\prod_{e\in f}\int d^2 n_{fe}\right)\left(\prod_f A_f^{BF}\right)\left(\prod_v A_v^{BF}\right)$$

where the sum/integral is over an irrep (ρ_f, k_f) of $SL(2,\mathbb{C})$ for each f, and for each $f \ni e$ a coherent state $|j_{fe}, n_{fe}\rangle \in \mathcal{H}_{\rho_f,k_f}$, $\vec{J}^2|j_{fe}, n_{fe}\rangle = j_{fe}(j_{fe}+1)$, $n_{fe}\cdot\vec{J}|j_{fe}, n_{fe}\rangle = j_{fe}|j_{fe}, n_{fe}\rangle$.

A_f^{BF} depends on (ρ_f, k_f) and A_v^{BF} on data associated to $f, e \ni v$.

II. Derivation of amplitude from idea of Plebanski: Gravity as constrained topological field theory.

- The data $\rho_f, k_f, j_{ef}, n_{ef}$ determine bivectors B^{IJ}_{ef} via $J^i_{ef} = \epsilon^i_{jk} B^{jk}_{ef}$, $K^i_{ef} = B^{0i}_{ef}$ with

$$\vec{J}^2_{ef} - \vec{K}^2_{ef} = \rho^2_f - k^2_f \qquad \vec{J}_{ef} \cdot \vec{K}_{ef} = \rho_f k_f \qquad \vec{J}_{ef} = j_f \vec{n}_{ef}$$

- Related to bivectors Σ_{ef} via $B_{ef} = \frac{1}{8\pi G}\left(\star\Sigma_{ef} + \frac{1}{\gamma}\Sigma_{ef}\right)$

- **"Linear simplicity"**:
  - For each 3-cell e^*, $\exists N^I_e$ **such that** $\Sigma^{IJ}_{ef}(N_e)_J = 0 \quad \forall f \ni e$.
  - Ensures all Σ_{ef} are **simple** and hence determine a **2-plane** $f^*(e)$ in Minkowski space.
  - Ensures that, for each 3-cell e^*, all $f^*(e)$ are in **same 3-plane** ($\perp N_e$).
  - When critical point equations hold in large spin limit, is equivalent to $\Sigma^{IJ}_{ef} = \int_{f^*(e)} e^I \wedge e^J$ for some constant e^I for each 3-cell e^*, and hence $B_{ef} = \frac{1}{8\pi G}\int_{f^*(e)}\left(\star e \wedge e + \frac{1}{\gamma} e \wedge e\right)$

- **Remains to quantize linear simplicity**

III. Resulting models: EPRL and generalizations

Restriction to simplicial complex with space-like triangles: EPRL

$$\rho_f/\gamma = k_f = j_{fe} \qquad \text{for all} \qquad e \in f$$

- Proposed along with Euclidean version by **E., Pereira, Rovelli, and Living in 2007**.
- Euclidean version for $\gamma < 1$ coincides with model by **Friedel and Krasnov** proposed earlier in **2007**.

General 2-complex with space-like triangles: KKL(-DHR)

- Even though the EPRL derivation of the above condition depended on cell-complex being simplicial, the condition itself does not! **It immediately generalizes to a general cell complex**.
- The exact same thing happens in the Euclidean signature.
- Was first noticed and proposed by **Kaminski, Kisielwoski, and Lewandowski** in the Euclidean signature in **2009**, and by **Ding, Han, and Rovelli** in the Lorentzian signature in **2010**.

III. Resulting models: EPRL and generalizations

Quantum simplicity with time-like triangles: Conrady and Hnybida 2010

Inclusion of cosmological constant: Haggard, Han, Kaminski, Riello 2014-2021

Restriction to a single orientation and removal of degenerate sector:
'proper vertex' E. and Zipfel 2011-2015. Related to later part of talk.

IV. Contributions of Jerzy and collaborators: To bring the covariant and canonical formulations of loop quantum gravity closer.

- **Spin foams and the Warsaw group**
  - Jerzy started doing research in spin foams after the EPRL and FK models.
  - He invited me to Warsaw in 2007 to give a week of day-long lectures on spin-foams to the quantum gravity group. I believe he invited other spin foam researchers to give similar series of lectures.
  - The Warsaw group – Jurek and others – made many important contributions to spin foams, and is still one of the few leading groups in spin-foams.

IV. Contributions of Jerzy and collaborators: To bring the covariant and canonical formulations of loop quantum gravity closer.

- **Kaminski, Kisielowski, and Lewandowski (KKL) generalization:**
  - EPRL uses simplicial complexes, so that spin-networks in the histories are always 4-valent. **In canonical LQG, such a restriction is not natural – all valences are allowed**.
  - KKL provides a very natural generalization of EPRL to include all cell complexes.
  - **has been central to spin foam cosmology** (Vidotto 2010-2011).
  - Kisielowski, Lewandowski, and Puchta (2011) also developed a **diagrammatic approach** to spin-foams that aides with **systematically categorizing all spin-foams** for given boundary graph, allowing the same authors in **(2012)** to systematically categorize all foams for Vidotto's dipole cosmology boundary states.
- **Kisielowski and Lewandowsk (2018):** Derived a **spin foam model coupled to a scalar field** starting from the canonical theory developed by **Domagala, Giesel, Kaminksi, and Lewandowski** (**2010** – "Gravity quantized" – which Kristina Giesel will talk about in the next talk.)

V. Classical limit of EPRL: Sum over local orientations and concern with equations of motion.

- Large spin limit of EPRL vertex amplitude for non-degenerate data:

$$A_v\left(\{\lambda j_f, \lambda n_{fe}\}\right) \sim \frac{C}{2}\left(e^{iS_{\text{Regge}}} + e^{-iS_{\text{Regge}}}\right)$$

  Here $\{n_{fe}\}_{f\in e}$ are the outward normals to the four triangles f^* for each tetrahedron e^*, and γj_f are their areas. That defines each tetrahedron in $\mathbb{R}^3$, which are then rotated in Minkowksi space to form each 4-simplex.

- Implies sum over local orientation: One orientation variable $\mu_\sigma \in \{-1, 1\}$ for each 4-simplex σ.

- In continuum limit:

$$S[g] = \int \mu(x) R(x) \sqrt{\text{det} g(x)} d^4x$$

V. Classical limit of EPRL: Sum over local orientations and concern with equations of motion.

- Equation of motion:

$$\delta S[g] = \int \mu\sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} d^4x + \int \mu\partial_\alpha\left(\tilde{V}^\alpha_{\mu\nu}\delta g^{\mu\nu} + \tilde{W}^{\alpha\beta}_{\mu\nu}\partial_\beta\delta g^{\mu\nu}\right) d^4x$$

$$= \int \mu\sqrt{g}G_{\mu\nu}\delta g^{\mu\nu} d^4x - \int (\partial_\alpha\mu)\,\tilde{V}^\alpha_{\mu\nu}\delta g^{\mu\nu} d^4x + \int \left(\partial_\beta\left(\tilde{W}^{\alpha\beta}_{\mu\nu}\partial_\alpha\mu\right)\right)\delta g^{\mu\nu} d^4x$$

$$= \int \mu\sqrt{g}\left(G_{\mu\nu}\delta g^{\mu\nu} - g^{-1/2}\mu\tilde{V}^\alpha_{\mu\nu}\partial_\alpha\mu + g^{-1/2}\mu\partial_\beta\left(\tilde{W}^{\alpha\beta}_{\mu\nu}\partial_\alpha\mu\right)\right)\delta g^{\mu\nu} d^4x$$

$\Rightarrow$ E.O.M. $G_{\mu\nu}$ can be distributional where μ changes sign!

Correct E.O.M. ($G_{\mu\nu} = 0$) only where μ is homogeneous!

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VI. Exactly analogous phenomenon in Ponzano-Regge, where equations of motion seem correct. **But are they?**

From spin to connection formulation: Manifest flatness

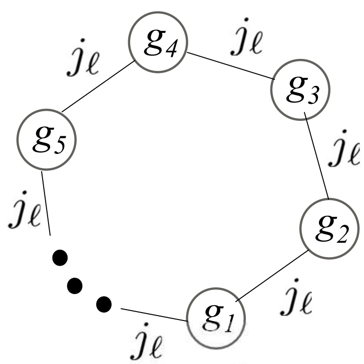
Given a 3D triangulation Δ with edges ℓ, triangles t, and tetrahedra σ,

$$W_{PR} = \sum_{\{j_\ell\}'} \prod_{\ell} (-1)^{2j_\ell}(2j_\ell + 1) \prod_{t} (-1)^{j_1+j_2+j_3} \prod_{\sigma} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$

N.B. $2j_\ell, j_1 + j_2 + j_3 \in \mathbb{N}$, so signs are well-defined!

where $\{j_\ell\}' := \{j_\ell\}_{\ell \in \mathrm{int}\Delta} \subset \mathbb{N}/2$. Assume, for simplicity, no boundary. Then

$$W_{PR} = \sum_{\{j_\ell\}} \int \left(\prod_t \mathrm{d}g_t \right) \prod_\ell (2j_\ell + 1)$$



$$= \int \left(\prod_t \mathrm{d}g_t \right) \prod_\ell \sum_j (2j + 1)\mathrm{Tr}_j(h_\ell)$$

$$= \boxed{\int \left(\prod_t \mathrm{d}g_t \right) \prod_\ell \delta(h_\ell)} \text{ “} = \int \mathcal{D}\omega \delta(F(\omega)) = \int \mathcal{D}\omega \mathcal{D}e \exp\left(i \int e \wedge F(\omega) \right) = \int \mathcal{D}\omega \mathcal{D}e \exp\left(iS[e, \omega]\right) \text{”}$$

Large spin asymptotics: Locally oriented Regge!

Setting $j_\ell = \lambda j_\ell^o$ ($\in \mathbb{N}/2$) for j_ℓ^o fixed.

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \underset{\lambda\to\infty}{\sim} \frac{1}{\sqrt{3\pi \mathrm{V}}} \cos\left(\sum_{a=1}^{6} j_a \Theta_a + \frac{\pi}{4} \right)$$

[Ponzano and Regge (1968); Dowdall, Gomes, and Hellmann (2009); Christodoulou, Långvik, Riello, Röken, and Rovelli (2012)]

where V is the volume of the tetrahedron with edge lengths λj_a and Θ_a is the *external* dihedral angle at edge a (angle between the normals to the two triangles at a).

implies

(This choice to express the -1's as exponentials is a generalization of that in Chistodoulou et al. and agrees for their triangulation.)

$$W_{PR} \sim \sum_{\{j_\ell\}}{}' \prod_\ell (e^{i\pi})^{2j_\ell}(2j_\ell+1) \prod_t (e^{-i\pi})^{j_1+j_2+j_3} \prod_\sigma \sum_{\mu_\sigma=\pm1} \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}} \exp i\mu_\sigma \left(\sum_{\ell\in\sigma} j_\ell \Theta_\ell(\sigma) + \frac{\pi}{4} \right)$$

$$= \sum_{\{j_\ell\}'} \sum_{\{\mu_\sigma\}} \left(\prod_\sigma \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}} \right) \exp i \left(S_{R,\mu} + \frac{\pi}{4} \sum_\sigma \mu_\sigma \right)$$

where

$$S_{R,\mu} := \sum_{\ell} j_\ell \left(\left(2 - |T_\ell| + \sum_{\sigma\in\Sigma_\ell} \mu_\sigma \right) \pi - \sum_{\sigma\in\Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) \right)$$

Here T_ℓ and Σ_ℓ respectively denote the set of triangles and tetrahedra containing ℓ, and $\theta_\ell(\sigma) = \pi - \Theta_\ell(\sigma)$ is the *internal* dihedral angle in σ at ℓ (angle inside σ between the planes of the two triangles at ℓ) **The sign μ_σ appearing here is the discrete analogue of sgn**$(\det(e))$..

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for $\mu_\sigma \equiv +1$

$$S_{R,+1} = \sum_{\ell\in \mathrm{int}\Delta} j_\ell \left(2\pi - \sum_{\sigma\in\Sigma_\ell} \theta_\ell(\sigma) \right) + \sum_{\ell\in\partial\Delta} j_\ell \left(\pi - \sum_{\sigma\in\Sigma_\ell} \theta_\ell(\sigma) \right) = S_{\mathrm{Regge}}$$

Exactly the Regge action, including correct boundary terms, for a general triangulation!

Equations of motion for fixed local orientations: Non-flatness!

Varying the internal j_ℓ:

$$\sum_{\sigma\in\Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) = \left(2 + \sum_{\sigma\in\Sigma_\ell} (\mu_\sigma - 1) \right) \pi$$

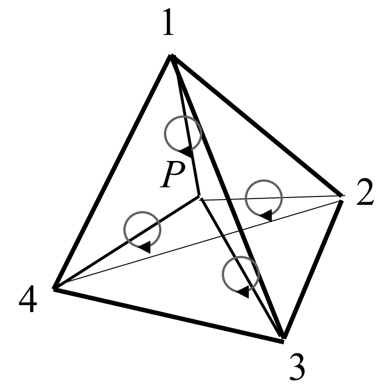
giving flatness, $\sum_{\sigma\in\Sigma_\ell} \theta_\ell(\sigma) = 2\pi$, **only** for $\mu \equiv 1$.

Simplest example: 4-1 Pachner move triangulation

For $\mu_1 = \mu_2 = \mu_3 = \mu_4 = +1$:

$$2\pi = \theta_{\ell_1}(\sigma_2) + \theta_{\ell_1}(\sigma_3) + \theta_{\ell_1}(\sigma_4)$$
$$2\pi = \theta_{\ell_2}(\sigma_1) + \theta_{\ell_2}(\sigma_3) + \theta_{\ell_2}(\sigma_4), \text{ etc.}$$

Flatness around all 4 internal ℓ_a, as expected.



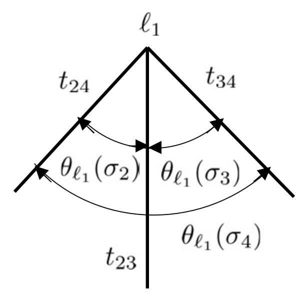
For $\mu_1 = \mu_2 = \mu_3 = +1$, $\mu_4 = -1$:

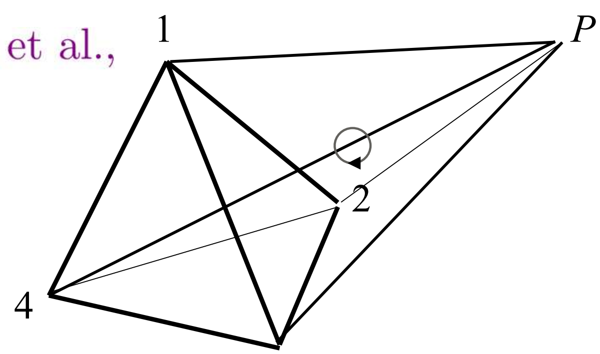
$$0 = \theta_{\ell_1}(\sigma_2) + \theta_{\ell_1}(\sigma_3) - \theta_{\ell_1}(\sigma_4)$$
$$0 = \theta_{\ell_2}(\sigma_1) + \theta_{\ell_2}(\sigma_3) - \theta_{\ell_2}(\sigma_4)$$
$$0 = \theta_{\ell_3}(\sigma_1) + \theta_{\ell_3}(\sigma_2) - \theta_{\ell_3}(\sigma_4)$$
$$2\pi = \theta_{\ell_4}(\sigma_1) + \theta_{\ell_4}(\sigma_2) + \theta_{\ell_4}(\sigma_3)$$

Flatness around ℓ_4,
but **not around** ℓ_1, ℓ_2, ℓ_3!

Following Christodoulou et al., we call this a **Spike**.

E.g., in plane $\perp \ell_1$:





Not flat!

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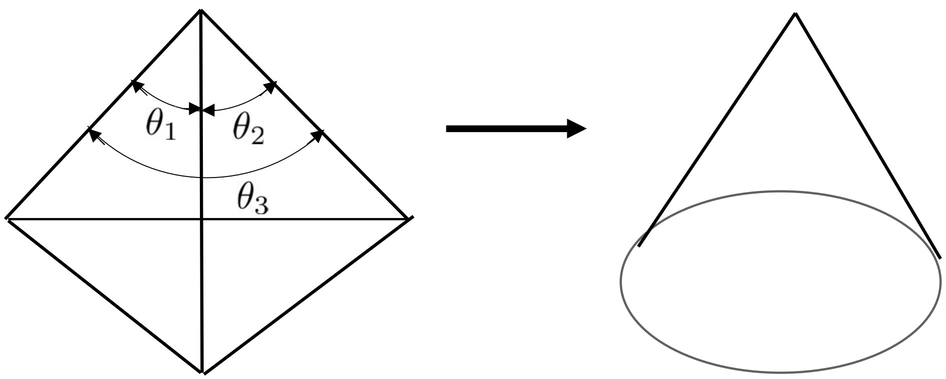
Equations of motion for fixed local orientations: Non-flatness!

2D analogue:

$$\theta_1 + \theta_2 = \theta_3$$



Conical singularity - not flat!

Key point: If interior dihedral angles around a hinge **don't sum to** 2π, then the geometry in a neighborhood of the hinge is not embeddable into $\mathbb{R}^n$ and so is **not flat!**

Flatness or curved spikes? Possible resolutions:

1. **Spikes generally correspond to bubbles for which model is ill-defined**

   - **In connection formulation**, Redundant δ's: **Divergence from too much flatness**
   - **In spin formulation**, unbounded sums over internal spins in spikes: **Divergence from too much curvature**

   Because both formulations are ill-defined in this case, there is no strict mathematical contradiction. Does we therefore give up learning from this paradox?

2. **Is the connection at spikes flat, even if geometry is not?**

   - Could $\sum_{\sigma\in\Sigma_\ell}\theta_\ell(\sigma) = \left(2 - \sum_{\sigma\in\Sigma_\ell}(\mu_\sigma - 1)\right)\pi$ somehow be the condition for flatness for the **spin-connection** determined by the **triad** e, which knows about orientation?
   - **Is the spin-connection even sensitive to the orientation of the triad?** Consider $\tilde{e}^i_a = \mu e^i_a$. Then

   $$\omega(\tilde{e})^{ij}_a = 2\tilde{e}^{b[i}\partial_{[a}\tilde{e}^{j]}_{b]} + \tilde{e}_{ak}\tilde{e}^{bi}\tilde{e}^{dj}\partial_{[d}\tilde{e}^k_{b]} = \cdots = 2\mu(\partial_b\mu)e^{b[j}e^{i]}_a + \omega(e)^{ij}_a$$

   In coordinate patch (x, y, z), if $\mu = \mathrm{sgn}(x)$, then $\mu\partial_b\mu = 2\mathrm{sgn}(x)\delta(x)\partial_b x = 0$ **if** we regularize $\mathrm{sgn}(x)$ symmetrically. Then $\omega(\mu e) = \omega(e)$, **so it seems ω is not sensitive to μ.**

Both exact flatness and arbitrarily curved spikes? Contradiction? Resolution?

2. **Is the connection at spikes flat, even if geometry is not?**

- One **can** define a **different discrete-only connection** g_t which **is sensitive to μ_σ**, with $\det g_t = -1$ when $\mu_\sigma = -\mu'_\sigma$ on either side of t. But then $g_t \in O(3)$, not $SU(2)$, so that is **not the connection here. What, then, is the connection here?**
- **Another possibility**: Connection and spin formulations of Ponzano-Regge are 'conjugate' to each other. As we saw, the amplitude imposes **exact flatness of connection with zero uncertainty**. **Does a generalized Heisenberg uncertainty relation then imply that uncertainty in the 'conjugate' curvature defined by spins is infinite? Would be consistent with the spikes.**

Analogous tension in continuum! Perhaps start here!

First order formulation

$$S[e,\omega] := \int e \wedge F(\omega) \quad \Rightarrow \text{E.O.M.}$$

- $d_\omega e = 0 \quad \Rightarrow \quad \omega = \omega(e)$
- $F(\omega) = 0$ **Flatness**

Second order formulation

$$S[e] := S[e,\omega(e)] = \int e \wedge F(\omega(e)) = \int \mu(x) R[g_{ab}] \sqrt{\det g(x)}\, d^3x =: S[g]$$

where $\mu(x) := \text{sgn}(\det(e(x)))$ and $g_{ab}(x) := e_a^i(x) e_{bi}(x)$.

$\Rightarrow$ By exact same derivation as in 4D, E.O.M. $G_{\mu\nu}$ can be distributional where μ changes sign!

Correct E.O.M. (flatness) only where μ is homogeneous!
How is that consistent with 1st order formulation?

Resolutions?

a) **Might resolution to paradox in 3D case also give insight to whether sum over orientations in 4D case is a problem?**

b) If it is a problem, should we `force' one homogeneous orientation?

i. `Proper' vertex [Engle, Zipfel 2012-2016], `causal'/`Feynman' spin-foam propagator [Livine, Oriti 2003,2004], **possibly related to** `causal evolution of spin networks' [Markopoulou, Smolin 1997]. **Fixing of time orientation?**

ii. Support from requiring projection onto kernel of Constraint operator in LQC [Ashtekar, Campiglia, Henderson 2010] and full LQG [Thiemann, Zipfel 2014].

iii. Modification to yield homogeneity of orientation at least in non-degenerate regions [Rovelli, Wilson-Ewing 2012]

Thank You!

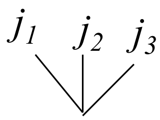
Extra Slides

From spin to connection formulation: Manifest flatness

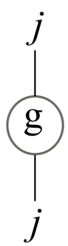
Diagrammatic notation elements:

$\rho_j(g) : V_j \to V_j$ denotes spin j irrep of $SU(2)$. $j \in \mathbb{N}/2,\ g \in SU(2)$.

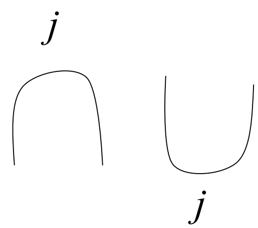
$$\dim\left(\mathrm{Inv}\left(V_{j_1} \otimes V_{j_2} \otimes V_{j_3}\right)\right) = \begin{cases} 1 & \text{if } j_1 + j_2 > j_3 \ \& \text{ cyclic and } j_1 + j_2 + j_3 \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$



denotes specific element of $\mathrm{Inv}\left(V_{j_1} \otimes V_{j_2} \otimes V_{j_3}\right)$ with phase convention chosen



denotes $\rho_j(g) : V_j \mapsto V_j$



bilinear form 'ϵ' on V_j, and its inverse, used to contract, raise and lower indices.

In spinorial realization $V_j = \{\psi^{A_1 \cdots A_{2j}} = \psi^{(A_1 \cdots A_{2j})}\}$, $\epsilon_{(A_1 \cdot A_{2j})(B_1 \cdots B_{2j})} = \epsilon_{A_1(B_1}\epsilon_{|A_1|B_1} \cdots \epsilon_{|A_{2j}|B_{2j})}$

The choice in writing signs as exponentials

In foregoing derivation,

- We made a choice to write $(-1)^{2j_\ell} = (e^{i\pi})^{2j_\ell}$ for each ℓ and $(-1)^{\sum_{\ell\in t} j_\ell} = (e^{-i\pi})^{\sum_{\ell\in t} j_\ell}$ for each t.
- If we had made the reverse choice $(-1)^{2j_\ell} = (e^{-i\pi})^{2j_\ell}$ and $(-1)^{\sum_{\ell\in t} j_\ell} = (e^{i\pi})^{\sum_{\ell\in t} j_\ell}$, then we would be led to an alternative action $\tilde{S}_{R,\mu}$ such that $\tilde{S}_{R,-} = -S_{\text{Regge}}$.
- The full ambiguity is **much broader**: A choice of $k_\ell, k_t \in 2\mathbb{N}+1$ at each ℓ and t, setting $(-1)^{2j_\ell} = (e^{ik_\ell\pi})^{2j_\ell}$ and $(-1)^{\sum_{\ell\in t} j_\ell} = (e^{ik_t\pi})^{\sum_{\ell\in t} j_\ell}$.
- Note this choice is just a choice of **how to write** the Ponzano-Regge amplitude. Thus, it **cannot affect the asymptotics** of Ponzano-Regge. Ponzano-Regge is a well-defined model and so has only one asymptotics!

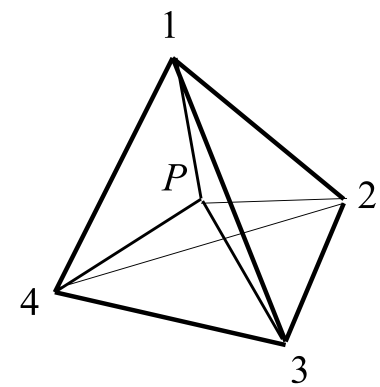
However,

- we next consider critical point equations from varying the j_ℓ's, which **makes sense only** if we **extend the action to continuous values** of the j_ℓ's, beyond $\mathbb{N}/2$.
- **This extension does depend** on the choice of how the signs are written as exponentials.
- Hence, the resulting **actions and critical point equations will depend** on this choice.
- Seems to **contradict** the fact that Ponzano-Regge, and hence its asymptotics, cannot depend on this choice. Nevertheless, as in the literature, we assume that the resulting asymptotics tell us **something heuristic** about Ponzano-Regge.

Equations of motion for fixed local orientations: Non-flatness!

Simplest triangulation with spike: 4-1 Pachner move ($^4\tau$ triangulation):

vertices:	4 boundary $a = 1, 2, 3, 4$ 1 internal P
tetrahedra:	4, σ_a, labeled by the vertex a not contained.
edges:	6 boundary ℓ_{ab} 4 internal $\ell_a := \ell_{aP}$
triangles:	4 boundary $t_a \in \sigma_a$ 6 internal $t_{ab} = \sigma_a \cap \sigma_b$



- $|T_{\ell_a}| = |T_{\ell_{ab}}| = 3$
- $\Sigma_{\ell_a} = \{\sigma_b\}_{b\neq a} \quad \Rightarrow \quad |\Sigma_{\ell_a}| = 3$

Critical point equations
from varying each internal spin j_ℓ:

$$\sum_{\sigma\in\Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) = \left(2 - |T_\ell| + \sum_{\sigma\in\Sigma_\ell} \mu_\sigma\right)\pi = \left(\sum_{\sigma\in\Sigma_\ell} \mu_\sigma - 1\right)\pi$$