Einstein-Maxwell gravitational instantons

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- Bernardo Araneda, MD. `arXiv: 25**.***`
- MD. Gravitational Instantons, old and new. `arXiv: 2501.00688`
- MD, Sean Hartnoll. Einstein-Maxwell gravitational instantons and five dimensional solitonic strings. CQG (2007).

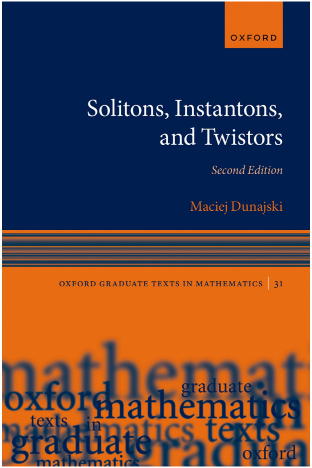
Jurek. Pune, December 1997

Jurek. Tux, February 2024

Instantons

- Solutions to the classical equations of motion (the Euler–Lagrange equations) in imaginary time and with finite action.
- In quantum field theory they give a leading quantum correction to the classical behaviour (WKB).
- Tunnelling behaviour between inequivalent vacua.
- ...no single phenomenon in physics has so far been attributed to instantons, and with no other explanation.
- Bridge between theoretical physics, and mathematics.
- Gravitational instantons: Semiclassical insight into quantum gravity. Only requires the validity of GR as a low energy theory. This talk: new gravitational instantons in Einstein–Maxwell theory.

Gravitational Instantons are solutions to the four-dimensional Einstein, or Einstein-Maxwell equations in Riemannian signature which give complete metrics and asymptotically ‘look-like’ flat space.



Euclidean Schwarzschild Metric

- Schwarzschild metric

$$g = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - \left(1 - \frac{2m}{r}\right) dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

  Removable singularity at $r = 2m$ (event horizon). Essential singularity at $r = 0$.

- Euclidean Schwarzschild metric. $t = i\tau, 2m < r < \infty$. Set $\rho = 4m\sqrt{1 - 2m/r}$. Near $\rho = 0$

$$g \sim d\rho^2 + \frac{\rho^2}{16m^2} d\tau^2 + 4m^2(d\theta^2 + \sin^2\theta d\phi^2).$$

  Identify $\tau \sim \tau + 8\pi m$. Flat and regular metric.

- Kerr black hole $\rightarrow$ Euclidean Kerr instanton.

Hawking's gravestone

ALF METRICS

- A complete regular four-dimensional Riemannian manifold (M, g) which solves the $\Lambda = 0$ Einstein or Einstein–Maxwell equations is called ALF (asymptotically locally flat) if it approaches S^1 bundle over S^2 at infinity.

$$\lim_{r\to\infty} g = (d\tau + 2n\cos\theta d\phi)^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2).$$

- Asymptotically flat (AF)= the S^1 bundle is trivial (so $n = 0$). E.g. Euclidean Schwarzschild and Euclidean Kerr.
- Lorentzian black hole uniqueness (Hawking, Carter, D. Robinson, . . .).
- Riemannian 'black hole uniqueness' conjecture: Euclidean Schwarzschild and Kerr are the only AF gravitational instantons.
- Doesn't extend to Einstein–Maxwell theory: there is more than Kerr–Newman (e. g. IWP instantons).

The Chen–Teo instanton

- Chen–Teo (2011, 2015) `arXiv:1107.0763, arXiv:1504.01235`: Riemannian 'black hole uniqueness' conjecture is wrong.
- Five parameter family of toric (two commuting Killing vectors) Riemannian Ricci flat metrics containing a two–parameter sub–family of AF gravitational instantons.
- One–sided type D (Aksteiner-Andersson 2024), Asymptotic charges (Kunduri-Lucietti 2021), Twistor Theory (MD-Tod 2024), tip of the iceberg (Li-Sun 2025).
- Question: Does there exist an Einstein–Maxwell AF analogue?

Some Einstein–Maxwell instantons

- Riemannian IWP metrics

$$g = \frac{1}{U\widetilde{U}}(d\tau + \omega)^2 + U\widetilde{U}d\mathbf{x}^2$$
$$F = U\widetilde{U} \star_3 d(U^{-1} + \widetilde{U}^{-1}) - \frac{1}{2}d(U^{-1} - \widetilde{U}^{-1}) \wedge (d\tau + \omega)$$

where $\nabla^2 U = \nabla^2 \widetilde{U} = 0,\ \nabla \times \omega = \widetilde{U}\nabla U - U\nabla\widetilde{U}$.

- Multi–centered solutions

$$U = \frac{4\pi}{\beta} + \sum_{m=1}^{N} \frac{a_m}{|\mathbf{x} - \mathbf{x}_m|}, \quad \widetilde{U} = \frac{4\pi}{\tilde{\beta}} + \sum_{n=1}^{\tilde{N}} \frac{\tilde{a}_n}{|\mathbf{x} - \tilde{\mathbf{x}}_n|},$$

- Regularity
  1. Lorentzian black holes iff $U = \tilde{U}$ (Hartle-Hawking 1972, Chruściel-Reall-Tod 2006)
  2. Riemannian instantons possible if $U \neq \tilde{U}$ (Whitt 1985, Yuille 1987, MD-Hartnoll 2007).

- Riemannian ALF: $\beta = \tilde{\beta}, \sum a_m - N = \sum \tilde{a}_n - \widetilde{N}$ and

$$U(\tilde{\mathbf{x}}_n)\tilde{a}_n = \widetilde{U}(\mathbf{x}_m)a_m = 1, \quad \forall m, n$$

  S^1–bundle over S^2 with $c_1 = N - \widetilde{N}$. AF if $N = \widetilde{N}$.

- Theorem (MD-Hartnoll 2007): Riemannian IWP are the most general Einstein–Maxwell instantons with super-covariantly constant Killing spinor. (Proof uses boundedness of $|F|^2 = |\nabla U^{-1}| + |\nabla \widetilde{U}^{-1}|$ and maximum principle.
- $N = 1, U = \widetilde{U}$: Extreme Reissner–Nordström instanton.
- $N > 1$. The only Ricci–flat limit is hyper–Kähler (U or $\widetilde{U}$ are constant). So Einstein–Maxwel Chen-Teo must be something else.

Einstein-Maxwell Chen-Teo

MD-Bernardo Araneda 2025:

- Three–parameter family (Q, ν, k) of toric (two commuting Killing vectors) one–sided type D Einstein–Maxwell instantons.
- Limiting cases: $Q = 0$: Chen-Teo (Ricci-flat). $\nu = -1$: 3–centre co-axial ALE Gibbons-Hawking, $\nu = 1 - 2Q$: charged Plebański–Demianski metrics (not regular).
- Asymptotic quantities: electric and magnetic charges, mass, angular momentum. Agree with Kunduri–Lucietti 2021 if $Q = 0$.
- Riemannian Harrison transformation of Chen–Teo Ricci–flat metrics.
- Conformal to Kähler: special $SU(\infty)$ Toda monopole.
- Reductions of anti–self–dual Yang–Mills equations with $SL(3, \mathbb{C})$ gauge group (twistor theory).
- Asymptotic quantities: electric and magnetic charges, mass, angular momentum. Agree with Kunduri–Lucietti 2021 if $Q = 0$.

DETAILS (MORE THAN YOU WISH FOR)

- A quartic f with four real roots. Set

$$
\begin{aligned}
f &= f(\xi) = a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 \\
\Phi &= f(x)y^2 - f(y)x^2 \\
H &= (\nu x + y)[(\nu x - y + Q(x+y))(a_1 - a_3 xy) \\
&\quad -(2(1-\nu) - 4Q)(a_0 - a_4 x^2 y^2)] \\
G &= [\nu^2 a_0 + 2\nu a_3 y^3 + (2\nu - 1)a_4 y^4 \\
&\quad -Q(\nu a_1 y + (2\nu-1)a_3 y^3 + 2(\nu-1)a_4 y^4)]f(x) \\
&\quad +[(1-2\nu)a_0 - 2\nu a_1 x - \nu^2 a_4 x^4 \\
&\quad +Q(2(\nu-1)a_0 + (2\nu-1)a_1 x + \nu a_3 x^3)]f(y) + a_2\nu^2\Phi.
\end{aligned}
$$

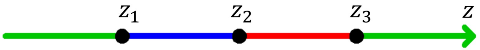
- Einstein–Maxwell metric

$$
g = \frac{kH}{(x-y)^3}\left(\frac{dx^2}{f(x)} - \frac{dy^2}{f(y)} - \frac{f(x)f(y)}{k\Phi}d\phi^2\right) + \frac{1}{\Phi H(x-y)}(\Phi d\tau + G d\phi)^2.
$$

- Torus action $K_i = \partial/\partial\phi^i$ where $\phi^i = (\phi, \tau)$,

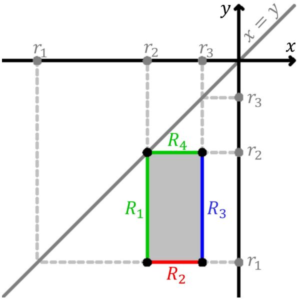
$$g = \Omega^2(dr^2 + dz^2) + G_{ij}d\phi^i d\phi^j, \quad i,j = 1,2$$

- $r^2 = \text{det}(G)$ and $*_2 dz = dr$.
- $\text{rank}(G(0,z)) = 1$ or 0 at *turning points* where K_i vanish.
- Rod structure



- Eliminate conical and orbifold singularities at the rods.
- Make sure that $\int_M |F|^2 \text{vol}_g < \infty$.
- 3–dimensional (Q, k, ν) moduli space of AF Einstein–Maxwell instantons on $M = \mathbb{CP}^2 \setminus S^1$.

Real roots of f: $r_1 < r_2 < r_3 < r_4$



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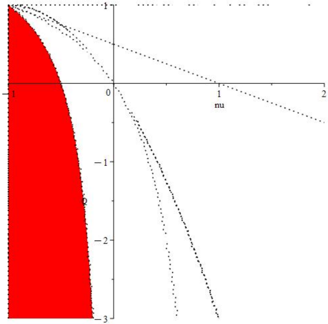

$$r_2 = -1, r_3 = \frac{(1-\nu)(1+r_1)Q - \nu^2 + 2\nu r_1 - 1}{(1+r_1)((\nu-1)Q - 2\nu)}, r_4 = 0$$

$$ {r_1}^2 + 2\frac{\nu(Q+\nu)}{Q\nu - 2\nu - 1} r_1 + \frac{\nu\left(Q\nu + \nu^2 - Q + 1\right)(Q+\nu)}{(Q\nu - 2\nu - 1)(Q\nu - Q - 2\nu)} = 0.$$

Outlook

- 'Modern view' on gravitational instantons: volume growth of a ball of large radius R: ALE: R^4, ALF: R^3, ALG: R^2, ALH: R or $R^{4/3}$.
- Very hard open problem: classify Ricci–flat or Einstein Maxwell instantons with given asymptotics.
- Hard open problem: classify toric AF gravitational instantons.
- Find an analogue of Chen-Teo with $\Lambda \neq 0$.

Thank You