Spin networks on quantum computers

Grzegorz Czelusta
Jagiellonian University, Cracow, Poland

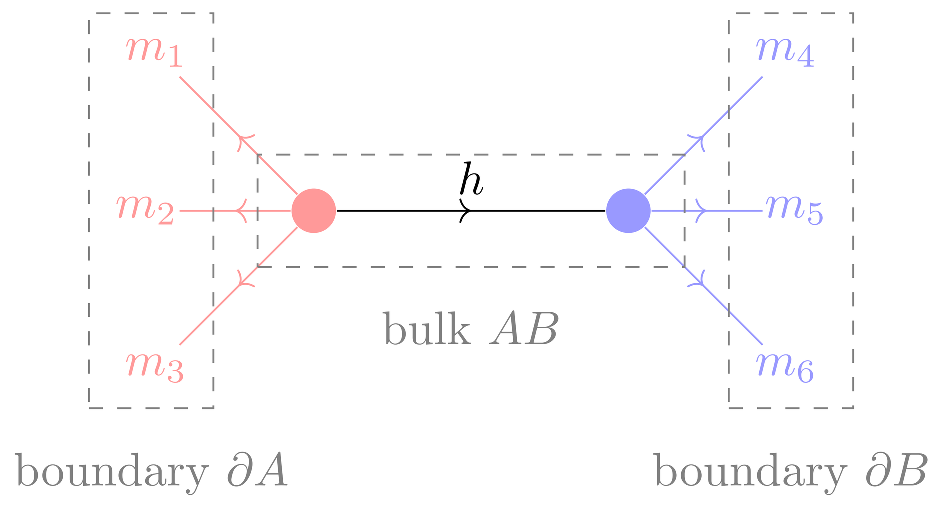
Quantum Cosmos Lab

Abstract

Spin network states are a powerful tool for building $SU(2)$ gauge theories on a graph. In loop quantum gravity (LQG), they have yielded many promising predictions, although progress has been limited by the computational challenge of dealing with high-dimensional Hilbert spaces. To explore more general configurations, quantum computing methods can be applied by representing spin network states as quantum circuits. We introduced a method for constructing quantum circuits for 4-valent Ising spin networks.

Spin networks

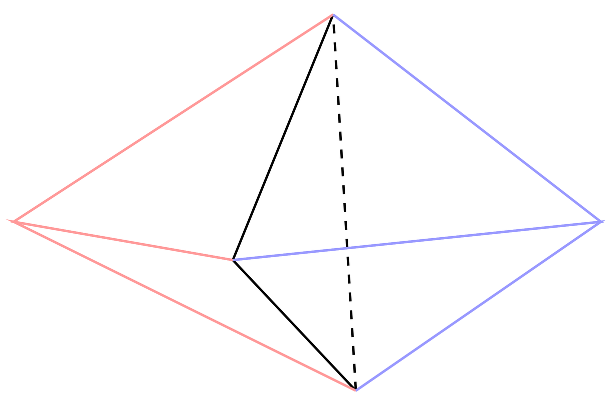
The Ising spin network are represented by graphs built out of four-valent nodes and the links (holonomies) associated with the fundamental ($j = \frac{1}{2}$) representations of the $SU(2)$ group.



The invariance with respect to the local gauge symmetry (imposed by the Gauss law) implies that the states $|\mathcal{I}\rangle$ at the nodes are spanned by the invariant, two-dimensional, spaces,

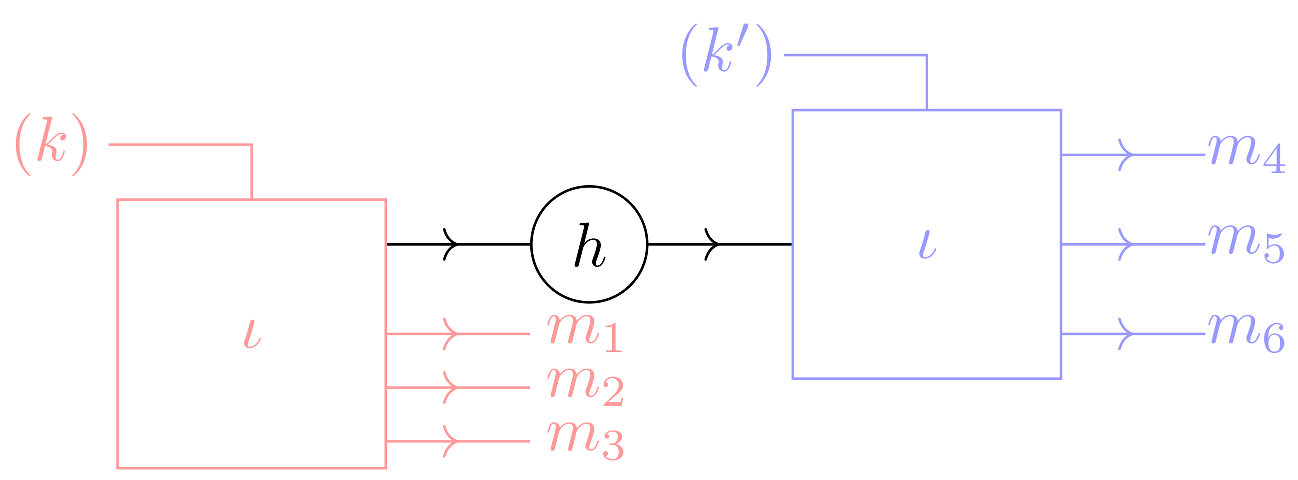
$$|\mathcal{I}\rangle \in \mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}\right). \tag{1}$$

Within LQG, the geometric operators (e.g., area or volume) have a clear geometric interpretation in terms of the spin network states. Specifically, 4-valent nodes are associated with quanta of volume, and the links describe their relative adjacency. An open spin network can be interpreted as a volume (bulk) enclosed by a surface (boundary). Example, shown above, is dual to two tetrahedra with one common face.



Spin networks as tensor networks

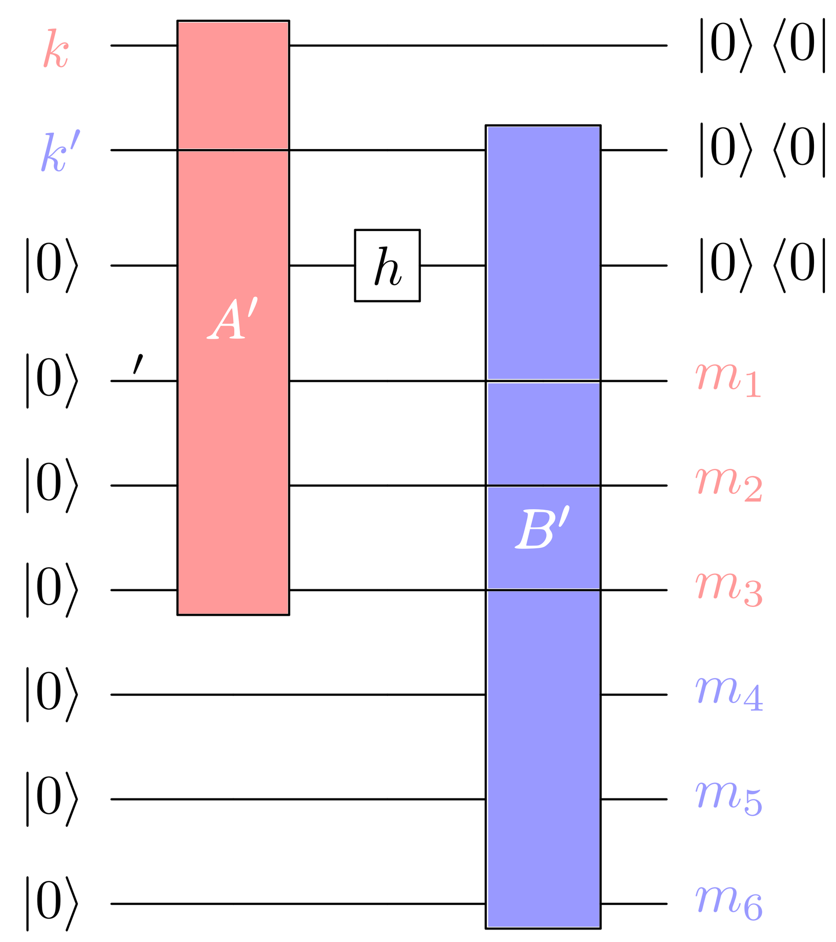
The state of a 4-valent node can be represented by a tensor with four magnetic indices (m_1, m_2, m_3, m_4) which correspond to links connected to this node and one more index (k) corresponding to the internal degree of freedom of the node. The positions of the indices are indicated by arrows, with the outgoing arrow corresponding to the upper index and the ingoing arrow corresponding to the lower index. The links of spin networks are labeled by $SU(2)$ holonomies, which can be represented as tensors using the Wigner $D(h)$ matrices.



$$\iota_{(k)}^{\ m_1m_2m_3m} D(h)^{m'}{}_{m}\, \iota_{(k')m'}{}^{m_4m_5m_6} \tag{2}$$

Quantum circuits

Tensor networks are used as an intermediate step in the process of translating spin networks into quantum circuits. Each tensor leg is two-dimensional and can be straightforwardly represented with a qubit. Additionally, every tensor can be presented as a quantum gate, employing ancillary qubits or measurements for nonunitary tensors. Also, each node possesses two internal degrees of freedom; hence, its state can be expressed with a single qubit.



Holographic bulk-boundary map

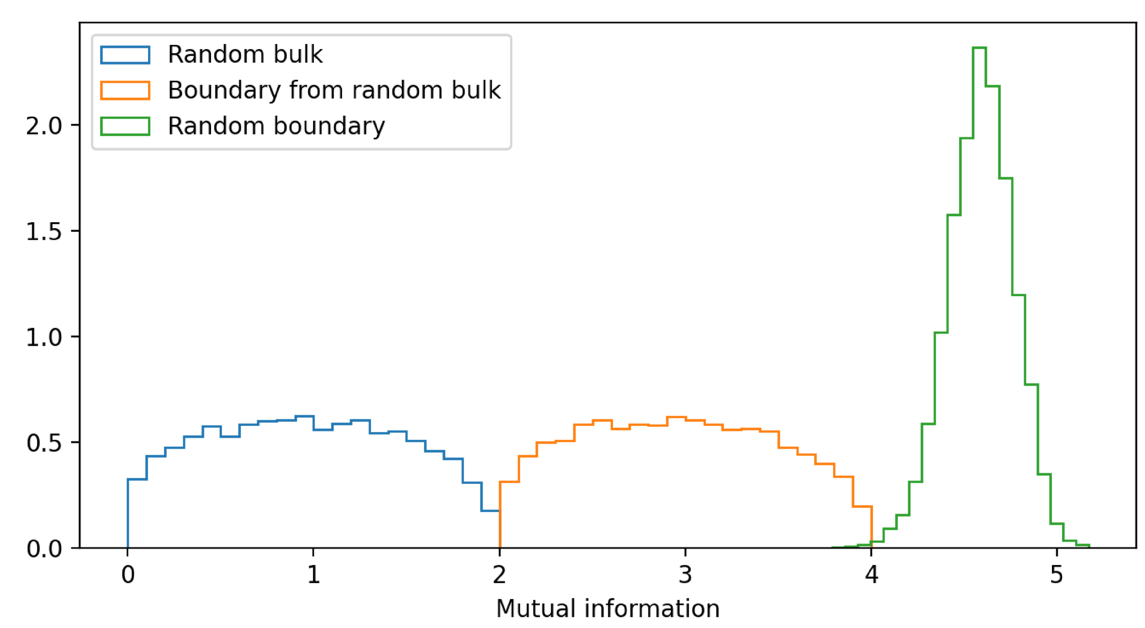
The bulk-boundary operator is an operator responsible for mapping the internal degrees of freedom of nodes (k) (bulk) to degrees of freedom of open links on the boundary (m). Such an operator can be practically realized using our scheme. Then we can examine its quantum information properties using methods from quantum computing. The folowing relation between entropy of bulk part and boundary part was shown [2]:

$$S(\hat{\rho}_{\partial A}) = S(\hat{\rho}_A) + \log(2j+1) \tag{3}$$

Using our circuit we can confirm these results measuring mutual information

$$I(\hat{\rho}_A, \hat{\rho}_B) := S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}). \tag{4}$$

between (m_1, m_2, m_3) and (m_4, m_5, m_6) parts of boundary for random bulk states.



We see that distributions blue and orange are shifted by 2, which is in agreement with Eq. (3) because for our case $j = \frac{1}{2}$ so

$$I(\hat{\rho}_{\partial A}, \hat{\rho}_{\partial B}) = I(\hat{\rho}_A, \hat{\rho}_B) + 2, \tag{5}$$

Moreover, we obtained also comparison between distribution of random boundary state and boundary state induces by state of a bulk (green and orange plots).

Bibliography

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