Symmetries of extremal horizons

Alex Colling

DAMTP, University of Cambridge

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- A. Colling, D. Katona, J. Lucietti. Rigidity of the extremal Kerr-Newman horizon. Lett. Math. Phys. **115**, 19 (2025).
- A. Colling. Symmetries of extremal horizons (work in progress).
- Work in progress with Jun Liu.

Motivation

Consider a D-dimensional analytic, stationary, asymptotically flat solution to the vacuum Einstein equations with a connected event horizon.

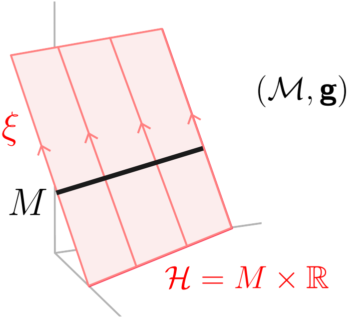
- No-hair theorem [Israel, Hawking, Carter, Robinson, ...]: for $D = 4$ the solution is a member of the Kerr family with parameters (M, J).
- Rigidity theorem [Hawking '72]: (i) the event horizon is a Killing horizon, and (ii) if the solution is rotating, it is also axisymmetric.

- Is there a quasi-local version of these results? $\Longrightarrow$ study of isolated horizons [Ashtekar, Lewandowski, ...].
- Extremal and non-extremal cases very different: in the extremal case the Einstein equations impose constraints involving only intrinsic data.

Extremal Killing horizons

Let $(\mathcal{M}, \mathbf{g}, \mathcal{T})$ be a spacetime of dim. $n+2$ containing an extremal Killing horizon $\mathcal{H}$ with generator ξ and compact cross section M.

$$\mathcal{L}_\xi \mathbf{g} = 0, \qquad \xi \perp \mathcal{H}, \qquad \mathrm{d}(\mathbf{g}(\xi,\xi)) \overset{\mathcal{H}}{=} 0.$$



We assume $(\mathcal{M}, \mathbf{g}, \mathcal{T})$ satisfies the Einstein equations (EE)

$$\mathrm{Ric}(\mathbf{g}) - \frac{1}{2} R_{\mathbf{g}}\, \mathbf{g} = \mathcal{T}.$$

Near-horizon equations

Definition

The near-horizon data (g, X, T, U) induced on M by $(\mathcal{M}, \mathbf{g}, \mathcal{T})$ consists of

- The induced Riemannian metric g on M.
- A 1-form $X \in \Omega^1(M)$ defined by

$$\mathrm{d}\xi \overset{\mathcal{H}}{=} \xi \wedge X.$$

- A symmetric $(0, 2)$ tensor T, the pullback of $\mathcal{T}$ to M.
- A function U on M defined by

$$\iota_\xi \mathcal{T} \overset{\mathcal{H}}{=} U\xi.$$

EE for $(\mathcal{M}, \mathbf{g}, \mathcal{T})$ imply near-horizon equations (NHE) for (M, g, X, T, U)

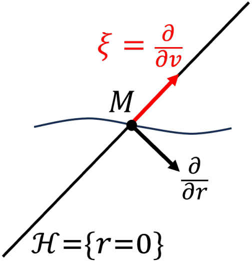
$$R_{ab} = \frac{1}{2} X_a X_b - \nabla_{(a} X_{b)} + T_{ab} - \frac{1}{n}(g^{cd} T_{cd} + 2U) g_{ab}.$$

Near-horizon geometry [Kunduri-Lucietti '13]

Given NH data, define the near-horizon geometry $(\mathbb{R}^2 \times M, \mathbf{g}_{\mathsf{NH}}, \mathcal{T}_{\mathsf{NH}})$

$$\mathbf{g}_{\mathsf{NH}} = 2\mathrm{d}v\mathrm{d}r + 2r\mathrm{d}v \odot X + r^2 F \mathrm{d}v^2 + g,$$
$$\mathcal{T}_{\mathsf{NH}} = 2U\mathrm{d}v\mathrm{d}r + 2r\mathrm{d}v \odot (\beta + UX) + r^2(\alpha + UF)\mathrm{d}v^2 + T.$$

$$F = \tfrac{1}{2}|X|^2 - \tfrac{1}{2}\nabla_a X^a + (1 - \tfrac{2}{n})U - \tfrac{1}{n} g^{ab} T_{ab},$$
$$\beta_a = -(\nabla^b - X^b)T_{ab} - UX_a,$$
$$\alpha = -\tfrac{1}{2}\nabla_a \beta^a + X^a \beta_a.$$



- EE for $(\mathbf{g}_{\mathsf{NH}}, \mathcal{T}_{\mathsf{NH}}) \iff$ NHE for (g, X, T, U).
- SIn is static if $\mathrm{d}X = 0$ and $\mathrm{d}F = XF$. It is rotating if X is not exact.

Outline

1. Rigidity theorem for extremal horizons
2. Examples: p-forms, scalars and non-abelian gauge fields
3. Symmetry enhancement of the near-horizon geometry
4. Classification of extremal horizons in Einstein-Maxwell theory

Rigidity theorem

In addition to compactness of M, we impose energy conditions

$$\text{For all null vectors } \ell,\ \mathcal{T}(\ell,\ell) \geq 0. \tag{EC1}$$

$$\text{For all null vectors } \ell,\ \mathcal{T}(\ell,\cdot) \text{ is causal.} \tag{EC2}$$

Extending results in [Dunajski-Lucietti '23, Colling-Katona-Lucietti '24]:

Theorem

Let (M, g, X, T, U) be a rotating solution to the near-horizon equations on a compact manifold M.

- If the associated near-horizon geometry satisfies the null energy condition (EC1), then (M, g) admits a Killing vector field K.
- If in addition the condition (EC2) holds, then K preserves the remaining near-horizon data (X, T, U).

Remark. If $(\mathbf{g}, \mathcal{T})$ satisfies (EC1) or (EC2), then so does $(\mathbf{g}_{\mathsf{NH}}, \mathcal{T}_{\mathsf{NH}})$.

Tensor identity

- K is constructed using an Ansatz. Given a smooth positive function Γ, define
$$K^\flat = \Gamma X + \nabla\Gamma.$$
- [Dunajski-Lucietti '23]: on compact M there exists a (unique up to scaling) choice of Γ s.t. $\nabla_a K^a = 0$.

Proposition

If (g, X, T, U) solves the NHE, there exist $\sigma \in (M)$ and $\tau \in C^\infty(M)$ s.t.
$$\frac{1}{4}|\mathcal{L}_K g|^2 + \gamma = \tau \nabla_a K^a + \nabla_a \sigma^a,$$
where
$$\gamma = T_{ab}K^a K^b - 2\Gamma K^a \beta_a - |K|^2 U + \Gamma^2 \alpha.$$

- We have $r^2\gamma = \mathcal{T}_{\mathsf{NH}}(\ell, \ell)$, null vector $\ell = \Gamma e_+ - \frac{1}{2\Gamma} r^2 |K|^2 e_- - r K^i e_i$.

Inheritance of symmetry

- Integrating tensor identity over M using (EC1) shows $\mathcal{L}_K g = \gamma = 0$.
- From $\mathcal{T}_{\text{NH}}(\ell, \ell) = 0$ and (EC2) we deduce $\mathcal{T}_{\text{NH}}(\ell, \cdot) \propto \ell$, giving

$$\Gamma\alpha = K^a\beta_a, \qquad \Gamma\beta_a + UK_a = K^b T_{ab.}$$

- It follows that $\mathcal{L}_K U = \mathcal{L}_K T = 0$. Proving $\mathcal{L}_K \Gamma = 0$ requires global argument using elliptic operator [Colling-Dunajski-Kunduri-Lucietti '24]

$$L\psi = -\Delta\psi + \nabla_a((\Gamma^{-1}\nabla^a\Gamma)\psi) + \Gamma^{-2}|K|^2\psi.$$

- Corollary [Kamiński-Lewandowski '24]: the following function A is constant

$$A = -\frac{|K|^2}{2\Gamma} + \frac{1}{2}\Delta\Gamma + (1 - \tfrac{2}{n})\Gamma U - \frac{1}{n}\Gamma g^{ab}T_{ab}.$$

Example: p-forms and uncharged scalars

Consider an $(n+2)$-dimensional theory with a p-form $\mathcal{F}$ and scalar Φ.

$$S = \int \left(R - \frac{1}{2}|\mathrm{d}\Phi|^2_{\mathbf{g}} - V(\Phi) - \frac{2}{p!}h(\Phi)|\mathcal{F}|^2_{\mathbf{g}} \right) \mathsf{vol}_{\mathbf{g}} + S_{\mathsf{top}}.$$

- On a cross section $i : M \to \mathcal{M}$ the matter induces a scalar ϕ, a closed p-form B and a $(p-2)$-form C by

$$\phi = i^*\Phi, \qquad B = i^*\mathcal{F}, \qquad \iota_\xi \mathcal{F} \overset{\mathcal{H}}{=} \xi \wedge C.$$

- We find $\gamma = \frac{1}{2}|\mathcal{L}_K\phi|^2 + \frac{2}{(p-1)!}h(\phi)|\iota_K B - \mathrm{d}(\Gamma C)|^2$, so if $h > 0$

$$\mathcal{L}_K\phi = 0, \qquad \iota_K B = \mathrm{d}(\Gamma C).$$

  Combine with matter equations to deduce $\mathcal{L}_K B = \mathcal{L}_K C = 0$.

- Can define near-horizon matter fields preserved by K

$$\Phi_{\mathsf{NH}} = \phi, \qquad \mathcal{F}_{\mathsf{NH}} = \mathrm{d}(-r\mathrm{d}v \wedge C) + B.$$

Non-abelian gauge fields and charged fields

Consider a gauge field $\mathcal{A}$ with gauge group G (compact & semisimple), curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$ and charged field Φ.

$$S = \int \left(R - \langle \mathcal{D}\Phi, \mathcal{D}\Phi \rangle - V(\Phi) - h(\Phi)\mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\right) \ \mathrm{vol}_{\mathbf{g}} + S_{\mathrm{top}}.$$

Here $\mathcal{D}\Phi = \mathrm{d}\Phi + \mathcal{A} \cdot \Phi$ is the covariant derivative.

- Induced matter data (ϕ, A, C) with curvature B and cov. deriv. D

$$\phi = i^*\Phi, \qquad A = i^*\mathcal{A}, \qquad \iota_\xi \mathcal{F} \stackrel{\mathcal{H}}{=} \xi \wedge C.$$

- γ contains extrinsic data $\psi = \partial_r \mathcal{D}_\xi \Phi$ and $H_a = \mathcal{F}_{ra}$ on $\mathcal{H}$ (in GNC).

$$\gamma = \langle D_K\phi - \Gamma\psi, D_K\phi - \Gamma\psi \rangle + 2h(\phi)\mathrm{Tr}\, |\iota_K B - D(\Gamma C) + \Gamma \mathcal{D}_\xi H|^2.$$

- Use matter equations to deduce [Li-Lucietti '13]

$$D_K\phi = -\Gamma C \cdot \phi, \quad \iota_K B = D(\Gamma C) \implies K \text{ preserves } (\phi, A, C)$$

Symmetry enhancement

Building on [Kunduri-Lucietti-Reall '07, Dunajski-Lucietti '23]:

Theorem

Any (extended) near-horizon geometry satisfying (EC1) and (EC2) with compact cross sections has isometry group containing the orientation-preserving isometry group of AdS_2, $\mathbb{R}^{1,1}$ or dS_2.

- Introducing coordinates x^i on M and ρ by $r = \Gamma(x)\rho$,

$$\mathbf{g}_{\text{NH}} = \Gamma[A\rho^2 \text{d}v^2 + 2\text{d}v\text{d}\rho] + g_{ab}(\text{d}x^a + K^a\rho\text{d}v)(\text{d}x^b + K^b\rho\text{d}v).$$

- The 2D metric in [ ] is AdS_2 if $A < 0$, $\mathbb{R}^{1,1}$ if $A = 0$ and dS_2 if $A > 0$. Isometries of [ ] extend to NHG (with 3D orbits if $K \neq 0$) when combined with appropriate shift in a coordinate χ along K.

- Isometries preserve EM tensor $\mathcal{T}_{\text{NH}}$ and near-horizon matter fields.

Special cases

- Example: extremal Kerr. $M = S^2$, $A = -\frac{1}{2a^2}$.

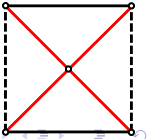
$$\mathbf{g}_{\mathsf{NH}} = \frac{1+x^2}{2}\left(-\frac{1}{2a^2}\rho^2 \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}\rho\right) + \frac{4a^2(1-x^2)}{1+x^2}\left(\mathrm{d}\phi + \frac{1}{2a^2}\rho \mathrm{d}v\right)^2 + \frac{a^2(1+x^2)}{1-x^2}\mathrm{d}x^2.$$

- If the sln is both static *and* rotating, there is a local isometric splitting

$$M = S^1 \times N, \qquad g = -A\Gamma \mathrm{d}\chi^2 + g_N, \qquad K = \partial_\chi$$

In this case the NHG is locally a warped product of AdS_3 and N.

- $A < 0$ for rotating solutions satisfying the strong energy condition. Doubly extremal horizons have $A = 0$, e.g. "ultracold" Reissner-Nordström-dS.



Four-dimensional Einstein-Maxwell theory

Einstein-Maxwell theory: induced data (g, K, Γ, B, C) satisfying NHE and

$$\mathrm{d}B = 0, \qquad \iota_K B = \mathrm{d}(\Gamma C), \qquad \nabla^a(\Gamma B_{ab}) = K_b C.$$

- 4 dimensions: complete classification (even with Λ!) using rigidity theorem as in the vacuum case [Dunajski-Lucietti '23]
  - Static case: g has constant curvature; $\Gamma, \star B, C$ are constant. [Chruściel-Tod '07, Kunduri-Lucietti '09, Kamiński-Lewandowski '24]
  - Axi-symmetric case: the Kerr-Newman horizon is the unique rotating solution admitting a $U(1)$ action preserving (g, X, B, C). [Lewandowski-Pawlowski '03, Kunduri-Lucietti '09]

Theorem [Colling-Katona-Lucietti '24]

Every rotating solution to the 4D Einstein-Maxwell NHE is given by an extremal Kerr-Newman horizon cross section.

Five dimensions

- 5D Einstein-Maxwell theory: classification incomplete even assuming $U(1) \times U(1)$ symmetry. Many known solutions; restrict to $M = S^3$. [Kunduri-Lucietti '09 '13, Hollands-Ishibashi '10, Blázquez-Salcedo Kunz Navarro-Lérida '13]
  - Vacuum: 3-parameter family of solutions, includes horizons of Myers-Perry and Kaluza-Klein black holes.
  - Static: 2-parameter family of solutions, with $U(1) \times U(1)$ symmetry and vanishing magnetic field $B = 0$.
  - Homogeneous: two 2-parameter families, $SU(2) \times U(1)$ symm.
- Can add Chern-Simons term $\propto \lambda \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A}$. "Charged Myers-Perry" found only for specific value of λ [Chong-Cvetic-Lu-Pope '05].
- [AC, Jun Liu]: explicit 3-parameter family (J_1, J_2, Q) interpolating between static solution and KK black hole for any λ.

New solutions

$M = S^3$, coordinates $y \in [0,1], \phi_{1,2} \in [0, 2\pi)$ and parameters (c_1, c_2, κ).

$$g = \frac{\Gamma}{4y(1-y)}\mathrm{d}y^2 + \frac{c_1^3\left[c_1 p_0(\kappa,\lambda) y + c_2 p_2(\kappa,\lambda)(1-y)\right]}{p_1(\kappa,\lambda)\,\Gamma}(1-y)\mathrm{d}\phi_1^2$$

$$+ \frac{2c_1^2 c_2^2 p_1(\kappa,\lambda)}{p_0(\kappa,\lambda)\Gamma} y(1-y)\mathrm{d}\phi_1\mathrm{d}\phi_2 + \frac{c_2^3\left[c_2 p_0(\kappa,\lambda)(1-y) + c_1 y p_2(\kappa,\lambda)\right]}{p_1(\kappa,\lambda)\,\Gamma} y\mathrm{d}\phi_2^2,$$

$$K = -\frac{2\kappa}{(\kappa+2\lambda)}\sqrt{\frac{p_3(\kappa,\lambda)}{(\kappa^2-1)}}\left(\frac{c_2}{c_1}\frac{\partial}{\partial\phi_1} + \frac{c_1}{c_2}\frac{\partial}{\partial\phi_2}\right),$$

$$\Gamma = \left[c_1 y + c_2(1-y)\right]^2, \qquad C = \frac{\sqrt{3c_1c_2}}{\Gamma}\sqrt{\frac{2(\kappa^2-1)}{p_0(\kappa,\lambda)}},$$

$$B = -\frac{\sqrt{3c_1c_2}}{\Gamma}\sqrt{\frac{p_3(\kappa,\lambda)}{2p_0(\kappa,\lambda)}}\,\mathrm{d}y \wedge \left(c_1^2\mathrm{d}\phi_1 - c_2^2\mathrm{d}\phi_2\right).$$

Entropy relations

- Angular momenta $J_i = J[m_i]$, charge Q and entropy $S = \frac{1}{4}\mathrm{Vol}_g(M)$ are accessible from horizon data.
- Integrating the constant $A = -\frac{|K|^2}{2\Gamma} + \frac{1}{2}\Delta\Gamma - \frac{4}{3}\Gamma C^2 - \frac{1}{3}\Gamma|B|^2$ leads to the entropy law [Hajian Seraj Sheikh-Jabarri '14]

$$\frac{A}{2\pi}S = \sum_i \omega^i J_i + \frac{4}{3}\mu Q,$$

where $K = \sum_i \omega^i m_i$ and $\mu = \Gamma C + \iota_K b$, $B = \mathrm{d}b$.

- In addition, for $\lambda = 0$ the two branches satisfy

$$S^{(1)} = \frac{4\sqrt{\pi}|Q^{(1)}|^{3/2}}{3^{3/4}} + \frac{3^{3/4}\pi^{3/2}}{4}\frac{|J_1^{(1)}J_2^{(1)}|}{|Q^{(1)}|^{3/2}}, \qquad S^{(2)} = 2\pi\sqrt{|J_1^{(2)}J_2^{(2)}|}.$$

Agrees exactly with numerical prediction [Horowitz-Santos '24]! But no Myers-Perry limit...

Summary

- Rigidity theorem: every rotating extremal horizon cross section in a theory satisfying e.g. the dominant energy condition admits a Killing field. There is a "degenerate surface gravity" A controlling the symmetry enhancement of the near-horizon geometry.
- Einstein-Maxwell theory: every rotating 4D solution to the Einstein-Maxwell NHE is given by the Kerr-Newman-(A)dS horizon. In 5D, many solutions are known, but probably more are missing.
- Open problems: – Charged Myers-Perry horizon? – 5D black holes containing new horizon solutions? – Horizon cross sections with only one Killing vector?

Thank you