Momentum space quantization for cotangent bundles of symmetric spaces and Coherent State transforms

José Mourão

(Dep. Mathematics, IST, Lisbon)

Lewandowski Memorial Conference

September 15–19 2025

On joint work mainly with:

T. Baier, A. Ferreira, J. Hilgert, W. Kirwin, J. P. Nunes, T. Thiemann and D. Wang

- I first met Jurek in the beginning of 1993 in Syracuse, when I visited Abhay Ashtekar.

- Then, in September 1993 started a sabbatical year in Penn State. Jurek was not there, because he was starting a postdoctoral position in Gainsville. That year was fantastic thanks to the excellent conditions and extremely friendly atmosphere that Abhay Ashtekar created for all of us.

- We started with Don Marolf working on trying to understand the support of the Ashtekar-Lewandowski measure. Got it wrong first but then corrected it in time to present the results in the Cornelius Lanczos International Centenary Conference, Chapel Hil, North Carolina, December, 1993.

- We were all five there, ALMMT, in the Lanczos Conference, and we started discussing.

- After Lanczos, Abhay and Jurek applied further the projective techniques to develop calculus and differential geometry on the space of quantum connections modulo gauge transformations.

- Sometime in the beginning of 1994 we came across a preprint (written in May, 1993) by Brian Hall, ”On the Segal-Bargman coherent state transform for compact Lie groups.”

- As a consequence we started a marathon to try to extend the results of Brian Hall’s paper to spaces of connections modulo gauge transformations.

- Four of us (AMMT) were in State College and one (L) in Gainsville, with just email, phone and fax machines.

- We would have long working meetings in State College and then would send our findings to Jurek.

- Very often he would tell us, yes, yes I know what you are telling me, but look what I did besides that ... Or Jurek would not agree with the path we had chosen and, as far as I remember, most of the times, we would agree that his path was best.

- This led to our journal of functional analysis paper, "Coherent state transforms for spaces of connections" submitted in December 1994.

- Most of my research since then was shaped by this paper and a key development by Thomas Thiemann, finding a complex canonical transformation mapping the $SU(2)$ spin connection to $SL(2,\mathbb{C})$–Ashtekar connection, "Reality conditions inducing transforms for quantum gauge field theory and quantum gravity", submitted in December, 1995.

I. Schrödinger quantization and quest for momentum space quantization

For systems with phase spaces given by cotangent bundles $M = T^*N$ with canonical symplectic form there is a natural quantization with quantum Hilbert space

$$\mathcal{H}^Q_{\mathsf{Sch}} = L^2(N, d\nu(x)) \,,$$

for some measure $d\nu$ on the configuration space N.

This is called the Schrödinger quantization and it is distinguished by the fact that classical observables that are functions of x alone (pullback of functions on N), $f_F(x,p) = F(x)$, act diagonally,

$$\widehat{f}_F = m_F = F \,.$$

These observables form a maximal abelian subalgebra of the (Poisson) algebra of observables and are the preferred observables of Schrödinger quantization.

If $N = \mathbb{R}^n$ or, more generally, $N = \mathbb{R}^{n-q} \times T^q$, where T^q is a q–dimensional torus, then the Schrödinger quantization of

$$M = T^*\mathbb{R}^{n-q} \times T^*T^q$$

is unitarily equivalent to the momentum quantization with wave functions given by (in general distributional) functions of momenta. In the momentum space quantization the maximal abelian subalgebra of preferred observables is that of functions of p alone.

Unitary equivalence of the two quantizations is given by the Fourier transform (discrete in the q directions of the torus).

For more general configuration manifolds N as eg symmetric spaces, $N = G/H$, the standard momentum space quantization of T^*G/H may not exist (or may not be interesting; In the case above N was an abelian Lie group and the momentum quantization corresponds to the decomposition of the quantum Hilbert space as direct integral of unitary irreducible representations of N).

An "interesting" momentum space quantization with quantum Hilbert space, $\mathcal{H}^Q_{\mathsf{mom}}$, should include a direct integral (summ) decomposition of $\mathcal{H}^Q_{\mathsf{mom}}$ into irreducible representations of G. This means that Casimir observables of the Hamiltonian G action (on T^*G/H) should be included in the preferred observables of any momentum space quantization and thus act diagonally. Such quantizations are then better called Fourier quantizations, $\mathcal{H}^Q_{\mathsf{F}}$.

II. Ambiguity of quantization and preferred observables

The dream of the founders of quantum mechanics was to have quantization as a well defined process assigning a quantum system to every classical system and satisfying the correspondence principle

$$\text{Quantization Functor (?)} : (M, \omega) \mapsto Q_\hbar(M, \omega) \overset{\hbar \to 0}{\mapsto} (M, \omega)$$

It was soon realized that this can never be the case even for the simplest systems.

Groenewold (1946) – van Hove (1951) no go Thm:

It is impossible, even for systems with one degree of freedom, to quantize all observables exactly as Dirac hoped

$$\begin{aligned} Q_\hbar(f) &= \widehat{f} \\ [Q_\hbar(f), Q_\hbar(h)] &= i\hbar\, Q_\hbar(\{f, g\}) \end{aligned}$$

and satisfy natural additional requirements like irreducibility of the quantization.

In order to quantize one needs to add additional data to the classical system. *eg* choose a (sufficiently big but not too big ...) (Lie) subalgebra of the algebra of all observables

$$\mathcal{A} = \mathrm{Span}_{\mathbb{C}}\{1, q, p\}$$

Then we have to study the dependence of the quantum theory on the additional data.

III. Geometric Quantization comes to the rescue

Fortunately Geometric Quantization allows to put (some) order in the apparent mess of this infinite dimensional family of different quantizations by introducing a "parametrization" in the space of quantizations of a fixed classical system.

Geometric quantization is mathematically perhaps the best defined quantization.

$$(M,\omega), \quad \frac{1}{2\pi\hbar}[\omega] \in H^2(M,\mathbb{Z})$$

Prequantum data: (L,∇,h), $L \to M, F_\nabla = \frac{\omega}{\hbar}$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M,L) = \overline{\left\{s \in \Gamma^\infty(M,L) : ||s||^2 = \int_M h(s,s)\ \frac{\omega^n}{n!} < \infty\right\}}$$

Pre-quantum observables:

$$\widehat{f} = Q_\hbar^{\mathrm{prQ}}(f) = \widehat{f}^{\mathrm{prQ}} = i\hbar\nabla_{X_f} + f$$

This almost works! But the Hilbert space is too large, the representation is reducible.
We need a smaller Hilbert space: Prequantization $\Rightarrow$ Quantization

Additional Data in Geometric Quantization

Generalizing what is done in the Schrödinger quantization, for systems with one degree of freedom, to fix a quantization one chooses (locally) a preferred observable – $F(q,p)$– and then works with wave functions of the form

$$\begin{aligned} \mathcal{H}^{prQ} \rightsquigarrow \mathcal{H}^{Q}_{\mathcal{P}_{\mathbf{F}}} &= \left\{ \Psi \in \mathcal{H}^{prQ} : \nabla_{X_F}\Psi = 0, ||\Psi|| < \infty \right\} = \\ &= \left\{ \Psi(q,p) = \psi(F)\, e^{-k(q,p)},\ ||\Psi|| < \infty \right\} \subset \mathcal{H}^{prQ} \end{aligned}$$

on which the preferred observable F and functions of it $u(F)$ act diagonally

$$Q^F_\hbar(u(F)) = \widehat{u(F)}^{\mathrm{prQ}}|_{\mathcal{H}^Q_{\mathcal{P}_{\mathbf{F}}}} = u(F).$$

For systems with n degrees of freedom one chooses (locally) n independent (possibly complex) observables in involution $F_1, \ldots, F_n$, $\{F_j, F_k\} = 0$. The polarization associated with this choice is the distribution

$$\mathcal{P}_{\mathbf{F}} = < X_{F_j}, j = 1, \ldots n > \ .$$

Equivalently, it is an integrable Lagrangian subbundle of the complexified tangent bundle of M.

$$\begin{aligned} \mathcal{H}^Q_{\mathcal{P}_{\mathbf{F}}} &= \left\{ \Psi \in \mathcal{H}^{prQ} \, : \, \nabla_{X_{F_j}} \Psi = 0, ||\Psi|| < \infty, j = 1, \ldots, n \right\} = \\ &= \left\{ \Psi(\mathbf{q}, \mathbf{p}) = \psi(F_1, \ldots, F_n) \, e^{-\, k(\mathbf{q},\mathbf{p})}, \, ||\Psi|| < \infty \right\} \subset \mathcal{H}^{prQ} \end{aligned}$$

(Non–)Equivalence of different Quantizations

Are all these quantizations (for different choices of F) physically equivalent?

NO!
Consider the observable: $H_\lambda = \frac{p^2}{2} + \frac{q^2}{2} + \lambda\frac{q^4}{4},\ \lambda \geq 0$
and let $Sp^{\mathsf{Sch}}(H_\lambda)$ denote the (discrete) spectrum of H_λ in the Schrödinger quantization, i.e. the spectrum of the operator

$$Q_\hbar^{\mathsf{Sch}}(H_\lambda) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{q^2}{2} + \lambda\frac{q^4}{4}$$

acting on $\mathcal{H}^Q_{\mathsf{Sch}} = \mathcal{H}^Q_{\mathcal{P}_{\mathsf{Sch}}} = L^2(\mathbb{R}, dq)$.

Now consider the 1−parameter family of quantizations with Hilbert spaces $\mathcal{H}^Q_{H_\lambda}$ for which the role of preferred observable is played by H_λ. Then, one finds that

$$\begin{aligned}
\mathcal{H}^Q_{\mathcal{P}_{H_\lambda}} &= \Big\{\Psi(q,p) : \nabla_{X_{H_\lambda}} \Psi = 0\Big\} = \\
&= \Big\{\Psi(q,p) = \psi(H_\lambda)\, e^{i\, G_\lambda(q,p)}\Big\} = \\
&= \left\{\sum_{n=0}^{\infty} \psi_n\, \delta(H_\lambda - E_n^\lambda)\, e^{iG_\lambda(q,p)}\right\}, \qquad (1)
\end{aligned}$$

where E_n^λ are defined by the Bohr-Sommerfeld conditions

$$\oint_{H_\lambda = E_n^\lambda} p dq = \hbar\,(n + \frac{1}{2}). \qquad (2)$$

Since H_λ acts diagonally on this quantization we conclude from (1) that its spectrum in this quantization is given by (2)

$$Sp^{H_\lambda}(H_\lambda) = \{E_n^\lambda, n \in \mathbb{N}_0\}$$

It is known that on one hand that for the harmonic oscillator, H_0,

$$Sp^{\mathsf{Sch}}(H_0) = Sp^{H_0}(H_0)$$

but on the other hand

$$Sp^{\mathsf{Sch}}(H_\lambda) \neq Sp^{H_\lambda}(H_\lambda)$$

for all $\lambda > 0$ so that the two quantizations $Q_\hbar^{\mathsf{Sch}}$ and $Q_\hbar^{X_{H_\lambda}}$ are physically inequivalent if $\lambda > 0$! Wins $Q_\hbar^{\mathsf{Sch}}$!

IV. Fourier and Momentum polarizations and quantizations

IV.1 Fourier Polarizations

Let (M, ω, G, μ) denote an Hamiltonian action of the connected Lie group G on the symplectic manifold (M, ω) with equivariant moment map

$$\mu : M \longrightarrow \mathfrak{g}^* .$$

Then if $\mathcal{P}_0$ is a G–invariant polarization geometric quantization defines a unitary representation of G on $\mathcal{H}_{\mathcal{P}_0}$. Consider its direct integral (summ) decomposition into irreducible representations π

$$\mathcal{H}^Q_{\mathcal{P}_0} = \int_{\widehat{G}} \mathcal{H}^{(\pi)}_{\mathcal{P}_0} \, d\widehat{\nu}(\pi) \tag{3}$$

Let $C_1, \ldots, C_r \in Z(\mathcal{U}(\mathfrak{g}))$ be independent Casimir generators seen as polynomial functions on $\mathfrak{g}^*$ and denote by $\widetilde{C}_j$ their μ pullback to M

$$\widetilde{C}_j = \mu^*(C_j) = C_j \circ \mu \,. \tag{4}$$

A particular class of G–invariant polarizations are the real or mixed polarizations $\mathcal{P}^{\mathsf{F}}$ which include the r directions (not necessarily independent) corresponding to the Casimir functions

$$X_{C_i} \in \Gamma_{\mathcal{P}^{\mathsf{F}}} \,, \qquad i = 1, \ldots, r \,.$$

This implies that $\mathcal{P}^{\mathsf{F}}$–polarized states have locally the form

$$\mathcal{H}^{Q}_{\mathcal{P}^{\mathsf{F}}} = \left\{ \widetilde{\psi}(\widetilde{C}_1, \ldots, \widetilde{C}_{r'}, y_1, \ldots, y_{n-r'})\, e^{i\alpha} \right\} , \tag{5}$$

where r' denotes the number of independent Casimir functions.

We have then two related consequences.

(i) The prequantum operators corresponding to $\widetilde{C}_j, j = 1, \ldots, r'$, act diagonally on $\mathcal{H}^Q_{\mathcal{P}^{\mathsf{F}}}$

$$\left(prQ(\widetilde{C}_j)(\widetilde{\psi})\right)(p) = \widetilde{C}_j\,\widetilde{\psi}(p) \tag{6}$$

where

$$p = (\widetilde{C}_1, \ldots, \widetilde{C}_{r'}, y_1, \ldots y_{n-r'})$$

(ii) From (5) and (6) the isotypical of $\mathcal{H}^Q_{\mathcal{P}^{\mathsf{F}}}$ is given by construction

$$\mathcal{H}^Q_{\mathcal{P}^{\mathsf{F}}} = \int_{\widehat{G}} \mathcal{H}^{(\pi)}_{\mathcal{P}^{\mathsf{F}}}\; d\widehat{\nu}(\pi)\,, \tag{7}$$

where

$$\mathcal{H}^{(\pi)}_{\mathcal{P}^{\mathsf{F}}} = \left\{\widetilde{\psi}(c_1^{(\pi)}, \ldots, c_{r'}^{(\pi)}, y_1, \ldots, y_{n-r'})\right\}$$

IV.2 Momentum Polarizations

Depending on (M, ω, G, μ), there are different possibilities of complementing

$$\langle X_{\widetilde{C}_1}, \ldots, X_{\widetilde{C}_{r'}} \rangle$$

to a Fourier polarization.

We call a G–invariant polarization a **momentum polarization** if it is Fourier and all the remaining polarized functions $y_1, \ldots, y_{n-r'}$ correspond to the pullback of functions from $\mathfrak{g}^*$.

In [BHKMN25] and [BFHMN25] we considered the cases $M = T^*(U)$ and $M = T^*(U/K)$, for U a compact connected Lie group and U/K a compact symmetric space.

The momentum polarizations we considered on $M = T^*(U) \cong T^*((U \times U)/U)$ and $M = T^*(U/K)$ correspond to complementing the Casimir functions with $U \times U$ and $U-$invariant complex structures on the coadjoint orbits in $\mu(T^*(U/K))$.

Example Let us consider the example

$$T^*(SU(2)/U(1)) \cong T^*(S^2)\,.$$

Polarized functions generating the momentum polarization

$$\begin{aligned} H &= ||\xi|| = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2} \\ w &= \frac{\xi_1+i\xi_2}{||\xi||-\xi_3} \end{aligned}$$

and we get

$$\mathcal{H}_{\mathcal{P}}^{\text{mom}} = \left\{\widetilde{\psi}(H,w)\,e^{i\alpha}\right\}$$

where

$$\widetilde{\psi}(H,w) = \sum_{n=1}^{\infty} c_n \delta(H-n-\frac{1}{2})\,p_n(w)\,.$$

and p_n are polynomials of degree n.

IV.3 Riemannian symmetric spaces of noncompact type

Of interest, also for the Geometric Langlands programme, is the case of Riemannian noncompact symmetric spaces, G/K (symmetric space dual of U/K), with G–semisimple and K its maximal compact subgroup,

$$M = T^*(G/K) \ .$$

(example to have in mind: $G = SL(n, \mathbb{R}), K = SO(n, \mathbb{R})$ and for the Iwasawa decomposition, $A =$ positive diagonal matrices with determinant 1 and N the upper triangular matrices with ones in the diagonal).

Iwasawa decomposition:

$$\begin{aligned} g &= k_1(g)\, e^{H(g)}\, n_1(g)\, (= \text{GS applied to columns}) \\ &= n_2(g)\, e^{A(g)}\, k_2(g)\, (= \text{GS applied to rows}) \end{aligned}$$

Use the invariant inner product on $\mathfrak{g}$ to identify $T^*(G/K)$ with $T(G/K)$. The moment map on $T(G/K)$ reads

$$\begin{aligned} \mu : T(G/K) \cong G \times_K \mathfrak{m} &\longrightarrow \mathfrak{g} \\ [g, u] &\mapsto Ad_g(u) \end{aligned}$$

The following result is known.

Theorem (W. Lisiecki) The fibers of the map

$$\begin{aligned} \psi : T_{\mathrm{reg}}(G/K) &\longrightarrow K/M \times \mathfrak{a}_+ \\ [g, H] &\longmapsto (k_1(g)M, H) \end{aligned} \tag{8}$$

define a momentum polarization with generating polarized functions

$$\mathcal{P}^{\mathrm{mom}} : \qquad \begin{aligned} F_\lambda([g, H]) &= \lambda(H) \\ G_f([g, H]) &= f(k_1(g)M) \end{aligned} \tag{9}$$

An important method to relate the quantizations corresponding to two polarizations consists in finding a one parameter group of (possibly complexified) canonical thansformations mapping one polarization to the other.

We have

Theorem (A Ferreira, J Hilgert, JM, JP Nunes)

$$\lim_{t \to \infty} (\varphi_t^{X_h})_* \left(\mathcal{P}^{\mathsf{Sch}}\right) = \mathcal{P}^{\mathsf{mom}} .$$

The next result, which are completing, shows that the appropriate lift of this theorem to the quantum bundle is (unlike the BKS pairing map) unitarily equivalent to the Fourier–Helgason transform.

Some of our papers on this general project of studying ambiguity in quantization via complex time Hamiltonian evolution and coherent state transforms:

BFHMN25] T Baier, AC Ferreira, J Hilgert, JM Mourão, JP Nunes, *Fibering polarizations and Mabuchi rays on symmetric spaces of compact type* Anal. Math. Phys. 15 (2025), 1-37.

BHKMN25] T Baier, J Hilgert, O Kaya, JM Mourão, JP Nunes, *Quantization in fibering polarizations, Mabuchi rays and geometric Peter-Weyl theorem*, Jour. Geom. Phys. 207 (2025), 105355.

MNPW24] JM Mourão, JP Nunes, A Pereira, D Wang, *A new look at quantization commutes with reduction in toric manifolds*, arXiv:2412.17157.

MMMN23] G Matos, B Mera, JM Mourão, PD Mourão, JP Nunes *Laughlin states change under large geometry deformations and imaginary time Hamiltonian dynamics*, Comm. Math. Phys. 399 (2023), 2045-2070.

[KMNT21] WD Kirwin, J Mourão, JP Nunes, T Thiemann, *Segal-Bargmann transforms from hyperbolic Hamiltonians* Jour. Math. Anal. App. 500 (2021), 125146.

[MN15] J.Mourão and J.P. Nunes, *On complexified analytic Hamiltonian flows and geodesics on the space of Kähler metrics*, Int Math Res. Not. (2015) 10624–10656.

[KMN13a] W. Kirwin, J.Mourão and J.P. Nunes, *Complex time evolution in geometric quantization and generalized coherent state transforms*, J. Funct. Anal. **265** (2013) 1460–1493.

[KMN13b] W. Kirwin, J.Mourão and J.P. Nunes, *Degeneration of Kaehler structures and half-form quantization of toric varieties*, Journ. Sympl. Geom. **11** (2013) 603–643.

[BFMN11] T. Baier, C. Florentino, J.Mourão and J.P. Nunes, *Toric Kahler Metrics Seen from Infinity, Quantization and Compact*

Tropical Amoebas, Journ. Differ. Geometry **89** (2011) 411–454.

Thank You my dear friend Jurek!

Thank You for all you have given to your family and friends, to science, to me