Hyperbolic Mass in $2 + 1$ Dimensions

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meeting in memory of Jurek Lewandowski
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joint work with Raphaela Wutte, arXiv:2401.04048
& with W. Cong, T. Queau, R. Wutte, arXiv:2411.07423

Why bother?

Why (2+1)-dim. general relativity with $\Lambda < 0$ interesting?

- AdS_3/CFT_2
- Interesting solutions (e.g. BTZ black holes)
- Mass is interesting

Outline

1. Solutions of Interest
2. Mass
3. Gluings of Solutions

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Peculiarities of 2+1

- All vacuum solutions to Einstein gravity at fixed cosmological constant are locally isometric

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,. \tag{1}$$

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- particle-like solutions and their quantisation with $\Lambda = 0$: Staruszkiewicz (1963), Deser-t'Hooft (1984), t'Hooft (1993)

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- unusual properties of energy with $\Lambda = 0$: Ashtekar and Varadarajan (1994)

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- unusual properties of energy with $\Lambda = 0$: Ashtekar and Varadarajan (1994)
- what about energy with $\Lambda < 0$?

Solutions of Interest

Known static solutions

1-parameter family of vacuum solutions [Bañados, Teitelboim, Zanelli, '92]

$$g_{2+1} = -(r^2 - m)dt^2 + \frac{dr^2}{r^2 - m} + r^2 d\varphi^2$$

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- many interesting quotients of 2-dim hyperbolic space are possible: compact, or with several locally asymptotically hyperbolic ends, or with cusps, or ...

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- the tensor field $\mu_{ij} = \mu_{ij}(\varphi)$ is called the mass aspect tensor

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- Write $\mu = \mu_{ij}\theta^i\theta^j$, where

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- The Hamiltonian mass *à la Kijowski-Tulczyjew* reads

$$H = \frac{1}{2\pi}\int_{\partial M}(\mu_{22} + 2\mu_{11})\, d\varphi =: \frac{1}{2\pi}\int_{\partial M}\mu\, d\varphi$$

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  - Example: $g = \frac{dr^2}{r^2 - m} + r^2 d\varphi^2\,,\; H = m$

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- Is H bounded from below by -1?

Hamiltonian charges and Witten's-type positivity proof

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Let $P_{ij} = K_{ij} - \mathrm{tr} K g_{ij}$, define the angular momentum aspect j

$$J := \frac{1}{2\pi}\int_{r=R} \underbrace{\lim_{R\to\infty} 2P^r{}_\varphi \sqrt{\det g}}_{=:j}\, d\varphi \tag{4}$$

$$C^1 := -\frac{1}{2\pi}\int_{S^1} \sin(\varphi)\,j\,d\varphi\,, \quad C^2 := \frac{1}{2\pi}\int_{S^1} \cos(\varphi)\,j\,d\varphi\,. \tag{5}$$

Mass in 2 Dimensions

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- [PTC, Cong, Queau, Wutte, '24]: the fact that two-dimensional ALH manifolds with only one asymptotic end carry exactly one spin structure, but with more than one asymptotic region or with interior boundaries carry two, has a surprising consequence:

Mass in 2 Dimensions

Theorem (PTC, Cong, Queau, Wutte, '24)

Let (M, g) be a smooth, complete Riemannian manifold, possibly with black-hole boundary, and suppose that (M, g, K) is ALH and satisfies the dominant energy condition. Then

$$H^0 + 1 \geq |J| + \sqrt{|\vec{C}|^2 + |\vec{H}|^2 + 2|\star(\vec{H} \wedge \vec{C})|}\,. \tag{6}$$

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If (M, g) carries a second spin structure (e.g., there is a black-hole boundary or another asymptotic end), then in addition to (6) it holds that

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But: are these objects well defined?

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Asymptotic Symmetries [Brown and Henneaux '86]

Acting with asymptotic symmetries

$$\varphi = f(\hat{\varphi}) - \frac{f''(\hat{\varphi})}{2\hat{r}^2}\,, \qquad r = \frac{\hat{r}}{f'(\hat{\varphi})}$$

with $f : S^1 \mapsto S^1$ any diffeomorphism, yields

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with new mass aspect function

$$\boxed{\hat{\mu} = \mu(f(\hat{\varphi}))f'(\hat{\varphi})^2 - 2S(f)(\hat{\varphi})\,, \quad S(f)(\hat{\varphi}) = \frac{f^{(3)}(\hat{\varphi})}{f'(\hat{\varphi})} - \frac{3}{2}\left(\frac{f''(\hat{\varphi})}{f'(\hat{\varphi})}\right)^2}$$

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For any μ, $\hat{H}$ can be made arbitrarily large.

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There exist functions μ which cannot be mapped to a constant by an asymptotic symmetry.

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$$H[\mu; f] := \int_{S^1} \left(\mu(f(\hat{\varphi}))f'(\hat{\varphi})^2 - 2S(f)(\hat{\varphi})\right) d\hat{\varphi}$$

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A mass aspect function μ is

1. good if $\underline{H}[\mu]$ is finite and attained;
2. bad if $\underline{H}[\mu]$ is - infinity;
3. ugly if $\underline{H}[\mu]$ is finite but not attained.

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For ugly functions $\underline{H}[\mu] = -1$.

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For good functions the infimum is attained on constants.

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For every μ there exists a vacuum metric without singularities near the conformal boundary.

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but elsewhere?

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  Remark: "Bad" has many flavors, which complicates a lot all relevant arguments.

Positive energy theorem revisited

Theorem (PTC, Cong, Queau, Wutte, '24)

Let (M, g) be a smooth, complete Riemannian manifold, possibly with black-hole boundary, and suppose that (M, g, K) is ALH and satisfies the dominant energy condition. Then the mass aspect function cannot be bad and

$$\underline{H}^0 + 1 \geq |\underline{J}| + \sqrt{|\underline{\vec{C}}|^2 + |\underline{\vec{H}}|^2 + 2|\star(\underline{\vec{H}} \wedge \underline{\vec{C}})|}\,. \tag{8}$$

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If (M, g) carries a second spin structure (e.g., there is a black-hole boundary or another asymptotic end), then in addition to (8) it holds that

$$\underline{H}^0 \geq |\underline{J}|\,. \qquad (9)$$

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Theorem (PTC, Wutte arXiv:2411.07423)

There exist static conformally compact ALH vacuum metrics with an ugly mass aspect function which are smooth except for one conical point.

Mass gap?

all good mass aspect functions are realised by

1. hyperbolic space $(m = -1)$,
2. together with the BTZ black holes $(m \in (0, \infty))$, or
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Mass gap?

all good mass aspect functions are realised by

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static, smooth, vacuum, ALH, complete: there is a mass gap

$$m \notin (-1, 0)$$

No mass gap in general

Recall the scalar constraint equation

$$R = \rho + 2\Lambda + |K|^2 - (\mathrm{tr} K)^2 \,,$$

where ρ is the matter density, and Λ is the cosmological constant, here normalised to -1.

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(boundaryless case).

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Gluing of initial data metrics

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what is the mass of the glued metric?

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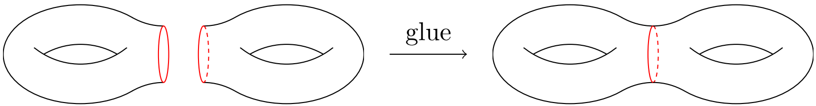
Theorem (PTC, Wutte arXiv:2411.07423)

There exist static conformally compact ALH vacuum metrics with an ugly mass aspect function which are smooth except for one conical point.

Gluing $n \geq 3$

Gluing Theorems in $n \geq 3$

[Isenberg, Lee & Stavrov 2010 (conformal); PTC & Delay 2015 (localised)]



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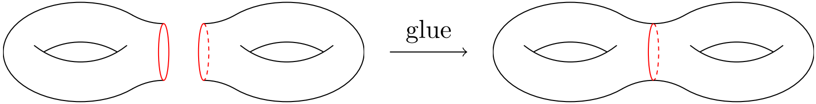

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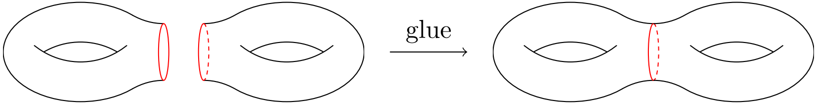

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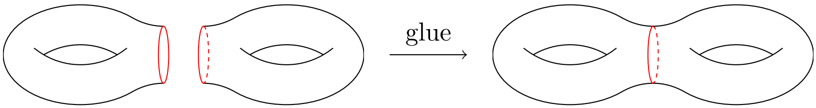

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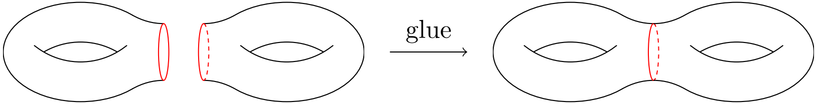

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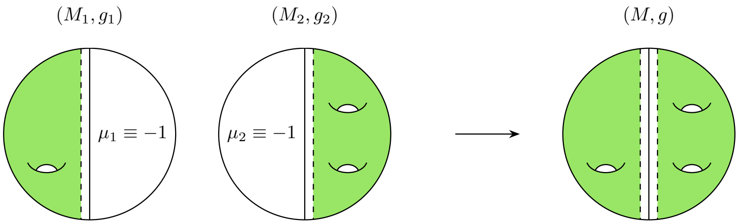
5. Cut and glue $\Rightarrow$ metric extends smoothly across gluing surface



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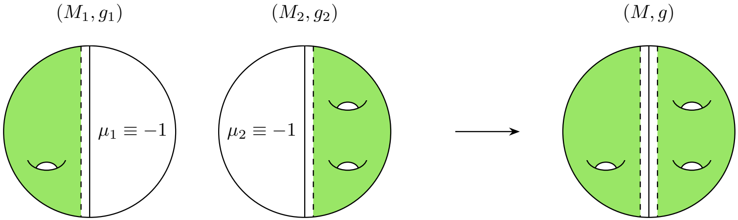

What is the mass of the resulting manifold?

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Theorem (Chruściel, Wutte, arXiv:2401.04048)

Given two asymptotically locally hyperbolic manifolds in dimension $n = 2$ with constant scalar curvature we have: If the initial masses m_1 and m_2 are positive, then the glued manifold has mass $m > 0$ equal to

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with gluing parameters $\omega_1 > 1$, $\omega_2 > 1$.

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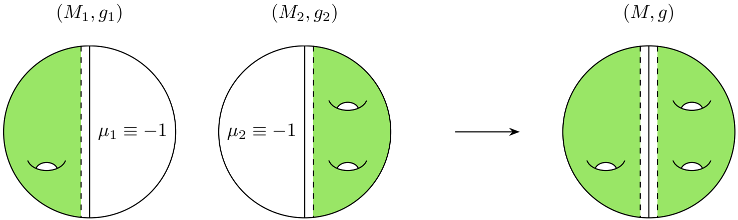

Idea of the proof: leverage [Balog, Feher, Palla 1997]

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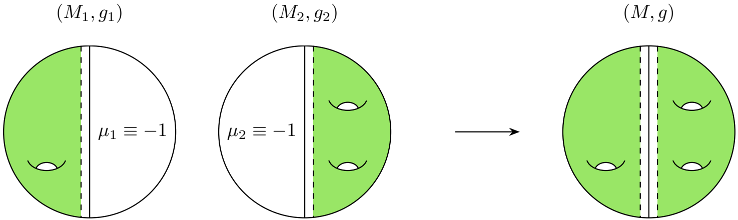

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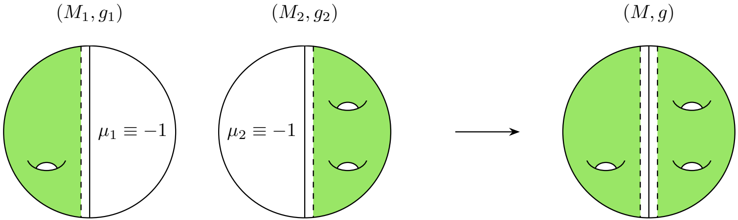

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- They classified functions $\mu : S^1 \to \mathbb{R}$ up to transformations $\mu \to \bar{\mu} = (f')^2 \, \mu \circ f - 2S(f)$ where $f \in \mathit{Diff}^+(S^1)$
- Classifying μ turns out to be equivalent to classifying $\mathit{Diff}^+(S^1)$-inequivalent solutions to the "Hill equation":

$$\frac{d^2\psi}{d^2\varphi} - \frac{\mu}{4}\psi = 0 \, .$$

Controlling mass: the trick (Balog, Feher, Palla)

Consider the Hill equation, with $\varphi \in \mathbb{R}$, where $\mu(\varphi)$ is 2π-periodic:

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The point is that under changes $\varphi \mapsto \bar{\varphi} = f(\varphi)$ the function

$$\bar{\psi}(\varphi) := \frac{\psi(f(\varphi))}{\sqrt{f'(\varphi)}}$$

satisfies again a Hill equation

$$\boxed{\frac{d^2\bar{\psi}}{d^2\varphi} - \frac{\bar{\mu}}{4}\bar{\psi} = 0\,,\ \text{with}\ \bar{\mu} = (f')^2\,\mu\circ f - 2S(f)\,.}$$

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The classification of the functions μ up to the transformation $\mu \mapsto \bar{\mu}$ can be derived from the invariants of the Hill equation.

Invariants of the Hill equation

$$\frac{d^2\psi}{d^2\varphi} - \frac{\mu}{4}\psi = 0\,, \quad \mu \text{ is } 2\pi\text{-periodic and } \varphi \in \mathbb{R}.$$

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$$\Psi(2\pi + \varphi) = \big(\psi_1(2\pi + \varphi), \psi_2(2\pi + \varphi)\big)$$

is also a basis of solutions. Hence there exists a matrix $\mathbf{M}$, called monodromy matrix, such that

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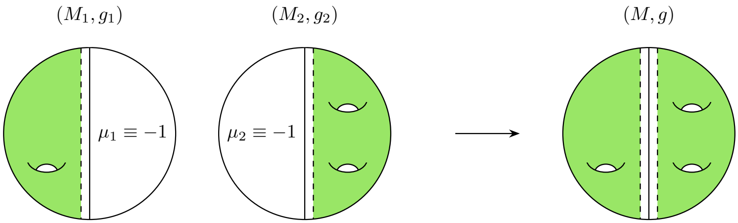
Trace of $\mathbf{M}$ is an invariant.

Back to our problem

Gluing $n = 2$, constant negative scalar curvature



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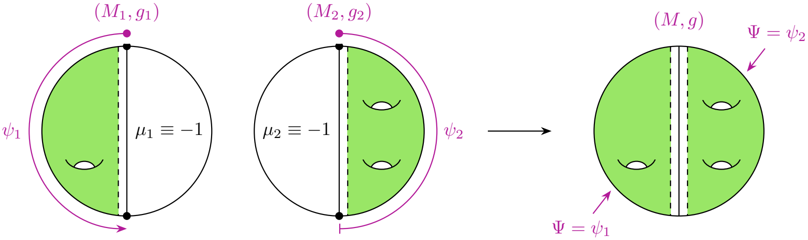

1. Start with
   (M_1, g_1), together with μ_1 and associated $(\Psi_1, \mathbf{M}_1)$
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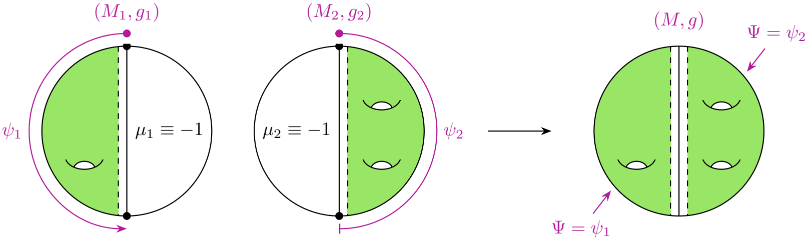

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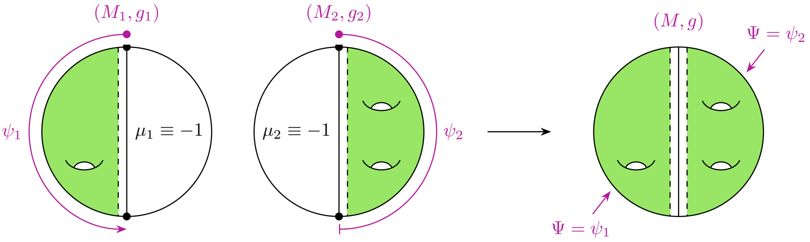
2. Glue the manifolds *and* associated Hill's functions Ψ_1 and Ψ_2
   $\Rightarrow (\Psi, \mathbf{M})$

Gluing $n = 2$



A calculation gives:

$$\mathbf{M} = -\mathbf{M}_1\mathbf{M}_2\,.$$

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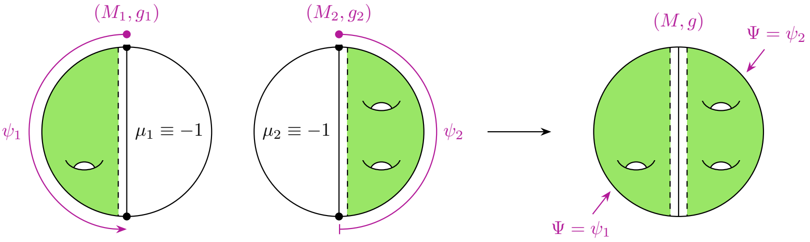

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Together with the zeros of the glued Hill functions Ψ this gives the mass of glued manifold by employing the classification result by [Balog, Feher, Palla '97].

Example result

Theorem (Chruściel, Wutte, arXiv:2401.04048)

Given two asymptotically locally hyperbolic manifolds in dimension $n = 2$ *with constant scalar curvature we have: If the initial masses* m_1 *and* m_2 *are positive, then the glued manifold has mass* m *determined from the equation*

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Recall: constant scalar curvature metrics provide vacuum time symmetric general relativistic initial data.

Proof: by gluing constant $\mu = m$ solutions, calculating the monodromy, and checking that one exhausts all cases of the classification by Balog, Feher and Palla.