Some reflections on Hamiltonians and GR

J. Fernando Barbero G.
Instituto de Estructura de la Materia, CSIC.
Jerzy Lewandowski Memorial Conference, September 17, 2025

Summary

back

- **Why Hamiltonians?**
- **Singular** Hamiltonian systems.
- **Dirac's vs. geometric (GNH)** approach.
- **A couple of examples:**
  - The Husain-Mehmood model.
  - The Euclidean self-dual action for GR.
- Concluding remarks.

Why Hamiltonians?

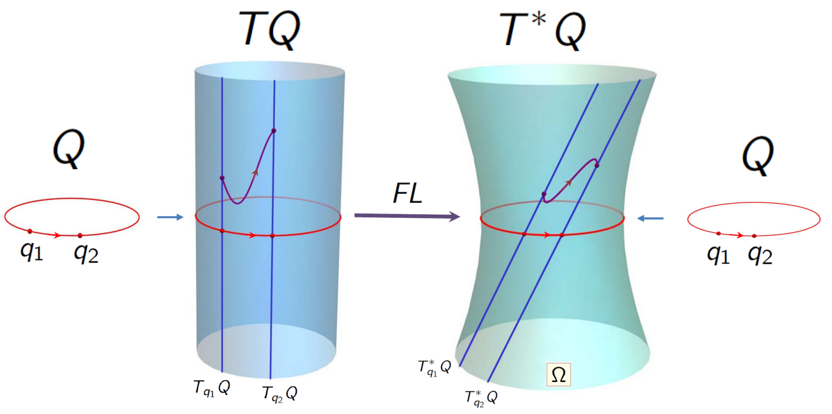
back

- **A frequent answer:** they are a suitable starting point for quantization. For instance, **loop quantum gravity** started as a program to implement Dirac's approach to quantization from a Hamiltonian description of GR in terms of $SU(2)$ connections.
- **They are also useful to understand the classical dynamics!**

A problem

The correct implementation of the methods designed to derive the Hamiltonian formulation for singular (a.k.a. constrained) systems is harder than it seems.

From Lagrangians to Hamiltonians back

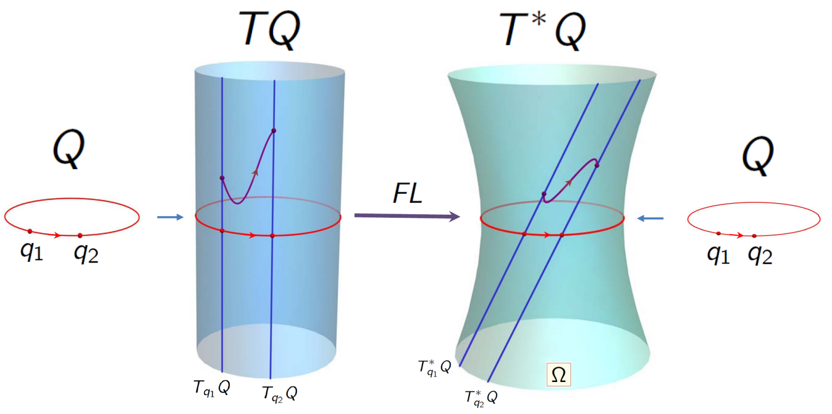


The fiber derivative (a.k.a. "the definition of momenta").

$$FL : TQ \to T^*Q : (q, v) \mapsto (q, \mathbf{p}) \,, q \in Q \,, \mathbf{p} \in T_q^*Q \,.$$

$$\mathbf{p}(w) := \frac{\mathrm{d}}{\mathrm{d}t} L(q, v + tw)\Big|_{t=0} \,, \quad v, w \in T_q Q \,.$$

From Lagrangians to Hamiltonians back

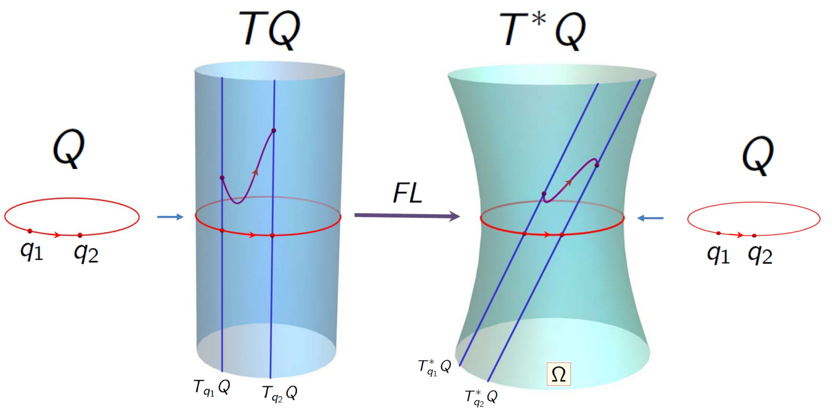


The energy: $E : TQ \to \mathbb{R} : (q, v) \mapsto \mathbf{p}(v) - L(q, v)\,.$

On solutions to the Euler-Lagrange equations **the energy is constant**, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} E\big(q(t), \dot{q}(t)\big) = 0,\ \forall t \in [t_1, t_2]\,.$$

From Lagrangians to Hamiltonians back



Another way to get the dynamics:

- Define the **Hamiltonian** $H : T^*Q \to \mathbb{R}$ as $H = E \circ FL^{-1}$.
- Find the **Hamiltonian vector field** X s.t. $\imath_X \Omega = \mathrm{d}H$.
- Get the **integral curves** of X and project them onto Q.

Singular Hamiltonian systems

back

- Those for which *FL* **is not a diffeo.**
- They can be studied by the traditional Dirac method or the **geometric approach** proposed by Gotay, Nester and Hinds (GNH).
- A crucial step in Dirac's method: **solve for the multipliers** introduced to define the total Hamiltonian. Their **arbitrary parts** give linear **combinations of primary constraints which are first class.** This is a time consuming task.
- The corresponding step in GNH is the **resolution of the equations for the components of the Hamiltonian vector fields and checking consistency of the dynamics.** Also a time consuming task.

Comparing Dirac and GNH

back

- **Dirac's** method uses the **whole phase space** T^*Q, whereas the **GNH** method is designed to work on the **primary constraint submanifold** M_0.
- Dirac uses the **canonical symplectic form**, whereas GNH employs its pullback onto M_0, which is **often degenerate**.
- Dirac's **total Hamiltonian** H_T is defined on the **whole phase space**, whereas the Hamiltonian in the GNH approach is defined **only on the primary constraint submanifold**.
- As a first approximation one can say that Dirac's method is **geared towards quantization** whereas GNH is **better suited to study the classical dynamics** but, in fact, both methods can be helpful for these goals.

The Husain-Mehmood model

back

V. Husain & H. Mehmood, PRD 109 (2024) 064016, arXiv:2312.06079
J.F.B.G., B. Díaz, J. Margalef-Bentabol & E.J.S. Villaseñor, arXiv:2507.12184

- Let Σ be a closed, orientable, 3-manifold (therefore parallelizable) and a 4-manifold $M = \mathbb{R} \times \Sigma$. Let us take the **action**

$$S_{\mathrm{HM}}(\Phi, \mathrm{A}) = \frac{1}{2} \int_{[\tau_1, \tau_2] \times \Sigma} \langle [\mathrm{d}_{\mathrm{A}} \Phi \wedge \mathrm{d}_{\mathrm{A}} \Phi] \wedge \mathrm{F}_{\mathrm{A}} \rangle$$

- The basic fields are $\mathfrak{g}$-valued forms $\Phi \in \Omega^0(M, \mathfrak{g})$ and $\mathrm{A} \in \Omega^1(M, \mathfrak{g})$. The symbol $\langle \cdot \wedge \cdot \rangle$ combines the exterior product of forms and a suitably $\mathfrak{g}$-invariant symmetric bilinear form. Also

$$\mathrm{F}_{\mathrm{A}} := \mathrm{dA} + \frac{1}{2} [\mathrm{A} \wedge \mathrm{A}]$$

$$\mathrm{d}_{\mathrm{A}} \mathrm{B} := \mathrm{dB} + [\mathrm{A} \wedge \mathrm{B}]$$

The symbol $[\cdot \wedge \cdot]$ combines the Lie bracket and the exterior product.

The Husain-Mehmood model

back

Comments:

- This is **closely related to the Husain-Kuchař model**. Its Hamiltonian analysis is quite interesting.
- In principle $\mathfrak{g}$ may be any finite-dimensional Lie algebra. In the following I will take $\mathfrak{g} = \mathfrak{su}(2)$, and use its (non-degenerate) **Cartan-Killing bilinear form**
- The **field equations** are:

$$[[\mathrm{F_A} \mathbin{\dot\wedge} \Phi] \mathbin{\dot\wedge} \mathrm{F_A}] = 0$$

$$\mathrm{d_A}[[\mathrm{F_A} \mathbin{\dot\wedge} \Phi] \mathbin{\dot\wedge} \Phi] = 0$$

- A good way to understand and disentangle the dynamics of this system is through its **Hamiltonian formulation**.

The HM model (Lagrangian)

back

- The **configuration space** for this field theory is

$$Q = \Omega^0(\Sigma, \mathfrak{g}) \times \Omega^1(\Sigma, \mathfrak{g}) \times \Omega^0(\Sigma, \mathfrak{g})$$

with points $(\phi, A, a) \in Q$ (two $\mathfrak{g}$-valued scalars and a 1-form).

- The **Lagrangian** is a real function $L : TQ \to \mathbb{R}$

$$L(v_q) = \int_\Sigma \Big(\langle [v_\phi \wedge \mathrm{d}_A \phi] \wedge F_A \rangle + \frac{1}{2} \langle v_A \wedge [\mathrm{d}_A \phi \wedge \mathrm{d}_A \phi] \rangle + \langle a \wedge \mathrm{d}_A [[F_A \wedge \phi] \wedge \phi] \rangle \Big)$$

TQ is trivial. We denote $v_q := \big((\phi, A, a), (v_\phi, v_A, v_a)\big) \in TQ$. The definitions of d_A and F_A are the expected ones.

The HM model (Hamiltonian) back

- The **phase space** T^*Q: with points $\mathrm{p}_q := ((\phi, A, a), (\mathrm{p}_\phi, \mathrm{p}_A, \mathrm{p}_a))$.
- **Momenta** (covectors acting on $w_q := ((\phi, A, a), (w_\phi, w_A, w_a)) \in TQ$)

$$\mathrm{p}_\phi(w_q) = \int_\Sigma \langle w_\phi \wedge [\mathrm{d}_A\phi \wedge F_A]\rangle \,,$$

$$\mathrm{p}_A(w_q) = \int_\Sigma \frac{1}{2}\langle w_A \wedge [\mathrm{d}_A\phi \wedge \mathrm{d}_A\phi]\rangle \,,$$

$$\mathrm{p}_a(w_q) = 0 \,.$$

- These conditions define the **primary constraint "submanifold"** M_0 where the dynamics unfolds.
- The **Hamiltonian** $H : \mathsf{M}_0 \to \mathbb{R}$ is

$$H(\mathrm{p}_q) = \int_\Sigma \langle a \wedge \mathrm{d}_A[\phi \wedge [F_A \wedge \phi]]\rangle$$

The HM model (Hamiltonian) back

- **Vector fields** $\mathsf{Y} = (\mathsf{Y}_\phi, \mathsf{Y}_A, \mathsf{Y}_a, \mathsf{Y}_{\mathrm{p}\phi}, \mathsf{Y}_{\mathrm{p}A}, \mathsf{Y}_{\mathrm{p}a}) \in \mathfrak{X}(T^*Q)$, where

$$\begin{aligned} \mathsf{Y}_\phi &: T^*Q \to \Omega^0(\Sigma, \mathfrak{g}), & \mathsf{Y}_{\mathrm{p}\phi} &: T^*Q \to \Omega^0(\Sigma, \mathfrak{g})^*, \\ \mathsf{Y}_A &: T^*Q \to \Omega^1(\Sigma, \mathfrak{g}), & \mathsf{Y}_{\mathrm{p}A} &: T^*Q \to \Omega^1(\Sigma, \mathfrak{g})^*, \\ \mathsf{Y}_a &: T^*Q \to \Omega^0(\Sigma, \mathfrak{g}), & \mathsf{Y}_{\mathrm{p}a} &: T^*Q \to \Omega^0(\Sigma, \mathfrak{g})^*, \end{aligned}$$

  $\mathsf{Y}_{\mathrm{p}\phi}$, $\mathsf{Y}_{\mathrm{p}A}$, $\mathsf{Y}_{\mathrm{p}a}$ are functions in T^*Q; "dual" because acting, respectively, on objects such as Y_ϕ, Y_A, and Y_a they give real functions in phase space.

- For vector fields **tangent** to M_0 we have

$$\mathsf{Y}_{\mathrm{p}\phi}(\cdot) = \int_\Sigma \big(\langle [\cdot \wedge \mathrm{d}_A \mathsf{Y}_\phi] \wedge F_A\rangle + \langle [\cdot \wedge [\mathsf{Y}_A \wedge \phi]] \wedge F_A \rangle + \langle [\cdot \wedge \mathrm{d}_A \phi] \wedge \mathrm{d}_A \mathsf{Y}_A\rangle\big)$$

$$\mathsf{Y}_{\mathrm{p}A}(\cdot) = \int_\Sigma \big(\langle \cdot \wedge [\mathrm{d}_A \mathsf{Y}_\phi \wedge \mathrm{d}_A \phi]\rangle + \langle \cdot \wedge [[\mathsf{Y}_A \wedge \phi] \wedge \mathrm{d}_A \phi]\rangle\big)$$

$$\mathsf{Y}_{\mathrm{p}a}(\cdot) = 0$$

The HM model (Hamiltonian)

back

- The **pullback** of $\mathrm{d}H$ acting on a vector field $\mathsf{Y}_0 \in \mathfrak{X}(\mathsf{M}_0)$ is

$$
\begin{aligned}
\mathrm{d}H(\mathsf{Y}_0) = \int_\Sigma \Big(& \langle \mathsf{Y}_\phi \wedge \big([\mathrm{d}_A a \wedge [F_A \wedge \phi]] + [F_A \wedge [\mathrm{d}_A a \wedge \phi]]\big)\rangle \\
& + \langle \mathsf{Y}_A \wedge \big([[\phi \wedge a] \wedge [F_A \wedge \phi]] - [\mathrm{d}_A \phi \wedge [\mathrm{d}_A a \wedge \phi]] \\
& \qquad + [\phi \wedge [[a \wedge \phi] \wedge F_A]] + [\phi \wedge [\mathrm{d}_A a \wedge \mathrm{d}_A \phi]]\big)\rangle \\
& + \langle \mathsf{Y}_a \wedge \mathrm{d}_A [\phi \wedge [F_A \wedge \phi]]\rangle \Big)
\end{aligned}
$$

- The **pullback** of Ω acting on a vector field $\mathsf{X}_0, \mathsf{Y}_0 \in \mathfrak{X}(\mathsf{M}_0)$ is

$$
\begin{aligned}
\Omega(\mathsf{X}_0, \mathsf{Y}_0) = \int_\Sigma \Big(& \langle \mathsf{Y}_\phi \wedge ([\mathsf{X}_A \wedge [F_A \wedge \phi]] + [F_A \wedge [\mathsf{X}_A \wedge \phi]])\rangle \\
& + \langle \mathsf{Y}_A \wedge ([\phi \wedge [\mathsf{X}_A \wedge \mathrm{d}_A \phi]] - [\mathrm{d}_A \phi \wedge [\mathsf{X}_A \wedge \phi]] \\
& \qquad - [\phi \wedge [\mathsf{X}_\phi \wedge F_A]] + [\mathsf{X}_\phi \wedge [F_A \wedge \phi]])\rangle \Big)
\end{aligned}
$$

The HM model (Hamiltonian)

back

- The **pullback** of $\mathrm{d}H$ acting on a vector field $\mathrm{Y}_0 \in \mathfrak{X}(\mathrm{M}_0)$ is

$$
\begin{aligned}
\mathrm{d}H(\mathrm{Y}_0) = \int_\Sigma \Big(& \langle \mathrm{Y}_\phi \wedge \big([\mathrm{d}_A a \wedge [F_A \wedge \phi]] + [F_A \wedge [\mathrm{d}_A a \wedge \phi]]\big)\rangle \\
& + \langle \mathrm{Y}_A \wedge \big([[\phi \wedge a] \wedge [F_A \wedge \phi]] - [\mathrm{d}_A \phi \wedge [\mathrm{d}_A a \wedge \phi]] \\
& \qquad + [\phi \wedge [[a \wedge \phi] \wedge F_A]] + [\phi \wedge [\mathrm{d}_A a \wedge \mathrm{d}_A \phi]]\big)\rangle \\
& + \langle \mathrm{Y}_a \wedge \mathrm{d}_A [\phi \wedge [F_A \wedge \phi]]\rangle \Big) \,.
\end{aligned}
$$

- The **pullback** of Ω acting on a vector field $\mathrm{X}_0, \mathrm{Y}_0 \in \mathfrak{X}(\mathrm{M}_0)$ is

$$
\begin{aligned}
\Omega(\mathrm{X}_0, \mathrm{Y}_0) = \int_\Sigma \Big(& \langle \mathrm{Y}_\phi \wedge ([\mathrm{X}_A \wedge [F_A \wedge \phi]] + [F_A \wedge [\mathrm{X}_A \wedge \phi]])\rangle \\
& + \langle \mathrm{Y}_A \wedge ([\phi \wedge [\mathrm{X}_A \wedge \mathrm{d}_A \phi]] - [\mathrm{d}_A \phi \wedge [\mathrm{X}_A \wedge \phi]] \\
& \qquad - [\phi \wedge [\mathrm{X}_\phi \wedge F_A]] + [\mathrm{X}_\phi \wedge [F_A \wedge \phi]])\rangle \Big) \,.
\end{aligned}
$$

The HM model (Hamiltonian) back

- The **pullback** of $\mathrm{d}H$ acting on a vector field $\mathsf{Y}_0 \in \mathfrak{X}(\mathsf{M}_0)$ is

$$\begin{aligned}\mathrm{d}H(\mathsf{Y}_0) = \int_\Sigma \Big(&\langle \mathsf{Y}_\phi \,\overset{\wedge}{,}\, \big([\mathrm{d}_A a \,\overset{\wedge}{,}\, [F_A \,\overset{\wedge}{,}\, \phi]] + [F_A \,\overset{\wedge}{,}\, [\mathrm{d}_A a \,\overset{\wedge}{,}\, \phi]] \big) \rangle \\ &+ \langle \mathsf{Y}_A \,\overset{\wedge}{,}\, \big([[\phi \,\overset{\wedge}{,}\, a] \,\overset{\wedge}{,}\, [F_A \,\overset{\wedge}{,}\, \phi]] - [\mathrm{d}_A \phi \,\overset{\wedge}{,}\, [\mathrm{d}_A a \,\overset{\wedge}{,}\, \phi]] \\ &\qquad + [\phi \,\overset{\wedge}{,}\, [[a \,\overset{\wedge}{,}\, \phi] \,\overset{\wedge}{,}\, F_A]] + [\phi \,\overset{\wedge}{,}\, [\mathrm{d}_A a \,\overset{\wedge}{,}\, \mathrm{d}_A \phi]] \big) \rangle \\ &+ \langle \mathsf{Y}_a \,\overset{\wedge}{,}\, \mathrm{d}_A [\phi \,\overset{\wedge}{,}\, [F_A \,\overset{\wedge}{,}\, \phi]] \rangle \Big) \,.\end{aligned}$$

- The **pullback** of Ω acting on a vector field $\mathsf{X}_0, \mathsf{Y}_0 \in \mathfrak{X}(\mathsf{M}_0)$ is

$$\begin{aligned}\Omega(\mathsf{X}_0, \mathsf{Y}_0) = \int_\Sigma \Big(&\langle \mathsf{Y}_\phi \,\overset{\wedge}{,}\, ([\mathsf{X}_A \,\overset{\wedge}{,}\, [F_A \,\overset{\wedge}{,}\, \phi]] + [F_A \,\overset{\wedge}{,}\, [\mathsf{X}_A \,\overset{\wedge}{,}\, \phi]]) \rangle \\ &+ \langle \mathsf{Y}_A \,\overset{\wedge}{,}\, ([\phi \,\overset{\wedge}{,}\, [\mathsf{X}_A \,\overset{\wedge}{,}\, \mathrm{d}_A \phi]] - [\mathrm{d}_A \phi \,\overset{\wedge}{,}\, [\mathsf{X}_A \,\overset{\wedge}{,}\, \phi]] \\ &\qquad - [\phi \,\overset{\wedge}{,}\, [\mathsf{X}_\phi \,\overset{\wedge}{,}\, F_A]] + [\mathsf{X}_\phi \,\overset{\wedge}{,}\, [F_A \,\overset{\wedge}{,}\, \phi]]) \rangle \Big) \,.\end{aligned}$$

The HM model (Hamiltonian) back

- The **pullback** of $\mathrm{d}H$ acting on a vector field $\mathsf{Y}_0 \in \mathfrak{X}(\mathsf{M}_0)$ is

$$
\begin{aligned}
\mathrm{d}H(\mathsf{Y}_0) = \int_\Sigma \Big(&\langle \mathsf{Y}_\phi \mathbin{\wedge\!\!\!\cdot} \big([\mathrm{d}_A a \mathbin{\wedge\!\!\!\cdot} [F_A \mathbin{\wedge\!\!\!\cdot} \phi]] + [F_A \mathbin{\wedge\!\!\!\cdot} [\mathrm{d}_A a \mathbin{\wedge\!\!\!\cdot} \phi]]\big)\rangle \\
&+ \langle \mathsf{Y}_A \mathbin{\wedge\!\!\!\cdot} \big([[\phi \mathbin{\wedge\!\!\!\cdot} a] \mathbin{\wedge\!\!\!\cdot} [F_A \mathbin{\wedge\!\!\!\cdot} \phi]] - [\mathrm{d}_A \phi \mathbin{\wedge\!\!\!\cdot} [\mathrm{d}_A a \mathbin{\wedge\!\!\!\cdot} \phi]] \\
&\qquad + [\phi \mathbin{\wedge\!\!\!\cdot} [[a \mathbin{\wedge\!\!\!\cdot} \phi] \mathbin{\wedge\!\!\!\cdot} F_A]] + [\phi \mathbin{\wedge\!\!\!\cdot} [\mathrm{d}_A a \mathbin{\wedge\!\!\!\cdot} \mathrm{d}_A \phi]]\big)\rangle \\
&+ \langle \mathsf{Y}_a \mathbin{\wedge\!\!\!\cdot} \mathrm{d}_A[\phi \mathbin{\wedge\!\!\!\cdot} [F_A \mathbin{\wedge\!\!\!\cdot} \phi]]\rangle \Big) .
\end{aligned}
$$

- The **pullback** of Ω acting on a vector field $\mathsf{X}_0, \mathsf{Y}_0 \in \mathfrak{X}(\mathsf{M}_0)$ is

$$
\begin{aligned}
\Omega(\mathsf{X}_0, \mathsf{Y}_0) = \int_\Sigma \Big(&\langle \mathsf{Y}_\phi \mathbin{\wedge\!\!\!\cdot} ([\mathsf{X}_A \mathbin{\wedge\!\!\!\cdot} [F_A \mathbin{\wedge\!\!\!\cdot} \phi]] + [F_A \mathbin{\wedge\!\!\!\cdot} [\mathsf{X}_A \mathbin{\wedge\!\!\!\cdot} \phi]])\rangle \\
&+ \langle \mathsf{Y}_A \mathbin{\wedge\!\!\!\cdot} ([\phi \mathbin{\wedge\!\!\!\cdot} [\mathsf{X}_A \mathbin{\wedge\!\!\!\cdot} \mathrm{d}_A \phi]] - [\mathrm{d}_A \phi \mathbin{\wedge\!\!\!\cdot} [\mathsf{X}_A \mathbin{\wedge\!\!\!\cdot} \phi]] \\
&\qquad - [\phi \mathbin{\wedge\!\!\!\cdot} [\mathsf{X}_\phi \mathbin{\wedge\!\!\!\cdot} F_A]] + [\mathsf{X}_\phi \mathbin{\wedge\!\!\!\cdot} [F_A \mathbin{\wedge\!\!\!\cdot} \phi]])\rangle \Big) .
\end{aligned}
$$

The HM model (Hamiltonian) back

Solving $\imath_{X_0}\Omega = dH$ for X_0

- **Equivalent to solving** $dH(Y_0) = \Omega(X_0, Y_0)$ for all $Y_0 \in \mathfrak{X}(M_0)$. Easy to do by **comparing term by term** the expressions in the previous slide.
- We find **secondary constraints:**

$$d_A[\phi \wedge [F_A \wedge \phi]] = 0$$

- and equations for the **components of the Hamiltonian vector field** X_0

$$[(X_A - d_A a) \wedge [F_A \wedge \phi]] + [F_A \wedge [(X_A - d_A a) \wedge \phi]] = 0$$

$$[(X_\phi - [\phi \wedge a]) \wedge [F_A \wedge \phi]] - [d_A\phi \wedge [(X_A - d_A a) \wedge \phi]]$$
$$-[\phi \wedge [(X_\phi - [\phi \wedge a]) \wedge F_A]] + [\phi \wedge [(X_A - d_A a) \wedge d_A\phi]] = 0$$

The HM model (Hamiltonian) back

- The vector fields found by solving the previous equation must be **tangent** to the set defined by the **secondary constraints**.

  Tangency condition

$$
\begin{aligned}
&\mathrm{d}_A[\mathrm{X}_\phi \mathbin{\dot\wedge} [F_A \mathbin{\dot\wedge} \phi]] + \mathrm{d}_A[\phi \mathbin{\dot\wedge} [\mathrm{d}_A \mathrm{X}_A \mathbin{\dot\wedge} \phi]] \\
&\qquad + \mathrm{d}_A[\phi \mathbin{\dot\wedge} [F_A \mathbin{\dot\wedge} \mathrm{X}_\phi]] + [\mathrm{X}_A \mathbin{\dot\wedge} [\phi \mathbin{\dot\wedge} [F_A \mathbin{\dot\wedge} \phi]]] = 0
\end{aligned}
$$

Comments:

- **Most of the work** goes into **solving the equations** for X_0 and checking **the tangency condition!**
- Although the equations for X_0 are **linear** it is quite difficult to find their solutions **in a usable form** (a surprisingly hard and interesting problem!)
- It is **crucial** to understand the behavior of these solutions **when the secondary constraints hold.**

The HM model (Hamiltonian) back

- Assuming that $\mathrm{d}_A\phi$ is a coframe, the components of the **Hamiltonian vector field** X_0 are:

$$
\begin{aligned}
&\mathsf{X}_\phi = \mathcal{L}_\xi \phi + [\phi \wedge (a - \xi \lrcorner A)]\,,\\
&\mathsf{X}_A = \mathcal{L}_\xi A + \mathrm{d}_A(a - \xi \lrcorner A) + \mathsf{N}\big(3\mathrm{d}\langle \phi \wedge \phi\rangle \phi - 2\langle \phi \wedge \phi\rangle \mathrm{d}_A\phi\big)\,,\\
&\mathsf{X}_a \quad \text{arbitrary}\,.
\end{aligned}
$$

  where
  - $\xi \in \mathfrak{X}(\Sigma)$ is an **arbitrary vector field**.
  - N is an **arbitrary smooth function** on T^*Q.
- We have:
  - **Diffeos** on Σ generated by ξ.
  - **Internal $SU(2)$ transformations** with gauge parameter $a - \xi \lrcorner A$.
  - **An additional gauge symmetry** associated with the arbitrary N.

The HM model (Hamiltonian)

back

Comments:

- The **tangency conditions** hold **on the submanifold defined by the secondary constraints.**
- The analysis has been performed under the hypothesis that the $\mathrm{d}_A\phi$ **define a coframe** on Σ.
- The local **gauge parameters** ξ and $a - \xi \lrcorner A$ are **arbitrary**.
- There **are extra gauge transformations** controlled by N. This is similar to GR in Ashtekar variables, where there are spatial diffeos, internal $SU(2)$ transformations and the non-trivial dynamics generated by the scalar constraint.

The Euclidean self-dual action for GR back

J.F.B.G., M. Basquens & E.J.S. Villaseñor, PRD109 (2024) 064047, arXiv:2312.12947

- **Basic fields:** $\mathrm{e}\,,\boldsymbol{\omega} \in \Omega^1(M,\mathfrak{su}(2))$, $\boldsymbol{\alpha} \in \Omega^1(M)$.
- $\boldsymbol{\alpha}$ and e chosen so that $\boldsymbol{\alpha}\otimes\boldsymbol{\alpha} + \langle \mathrm{e}\otimes\mathrm{e}\rangle$ is a **Euclidean metric**. As a consequence $(\boldsymbol{\alpha},\mathrm{e})$ defines a **non-degenerate tetrad**.
- **Covariant exterior differential** $\mathbf{D}$:

$$\mathbf{D}\mathrm{e} := \mathbf{d}\mathrm{e} + [\boldsymbol{\omega} \,\dot{\wedge}\, \mathrm{e}]$$

Curvature 2-form:

$$\mathbf{F} := \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \,\dot{\wedge}\, \boldsymbol{\omega}]$$

- The **Euclidean self-dual action** for General Relativity is

$$S(\mathrm{e},\boldsymbol{\omega},\boldsymbol{\alpha}) := \int_{[\tau_1,\tau_2]\times\Sigma} \left(\frac{1}{2}\langle[\mathrm{e}\,\dot{\wedge}\,\mathrm{e}]\,\dot{\wedge}\,\mathbf{F}\rangle - \boldsymbol{\alpha}\wedge\langle\mathrm{e}\,\dot{\wedge}\,\mathbf{F}\rangle\right)$$

The Euclidean self-dual action for GR back

- The first term is the **Husain-Kuchař action**.
- The action is **invariant** under the $SU(2)$ **gauge transformations**:

$$\begin{aligned} \delta_1 \boldsymbol{\omega} &= \mathbf{D\Lambda} \\ \delta_1 \boldsymbol{\alpha} &= 0 \qquad\qquad \boldsymbol{\Lambda} \in \Omega^0(M, \mathfrak{su}(2)) \\ \delta_1 \mathbf{e} &= [\mathbf{e} \,\dot{\wedge}\, \boldsymbol{\Lambda}] \end{aligned}$$

- The action is also **invariant** under the **extra** $SU(2)$ **transformations**

$$\begin{aligned} \delta_2 \boldsymbol{\omega} &= 0 \\ \delta_2 \boldsymbol{\alpha} &= \langle \boldsymbol{\Upsilon} \,\dot{\wedge}\, \mathbf{e} \rangle \qquad\qquad \boldsymbol{\Upsilon} \in \Omega^0(M, \mathfrak{su}(2)) \\ \delta_2 \mathbf{e} &= -\boldsymbol{\Upsilon}\boldsymbol{\alpha} + [\mathbf{e} \,\dot{\wedge}\, \boldsymbol{\Upsilon}] \end{aligned}$$

- δ_1 and δ_2 are **independent** but **do not commute**. Some linear combinations of them do commute. Full symmetry: $SU(2) \otimes SU(2)$.

The Euclidean self-dual action for GR back

- The **field equations** are:

$$\mathbf{D}(\boldsymbol{\alpha} \wedge \mathbf{e}) + [\mathbf{e} \overset{\wedge}{,} \mathbf{De}] = 0$$

$$\boldsymbol{\alpha} \wedge \mathbf{F} + [\mathbf{e} \overset{\wedge}{,} \mathbf{F}] = 0$$

$$\langle \mathbf{e} \overset{\wedge}{,} \mathbf{F} \rangle = 0$$

They are **equivalent to the Euclidean Einstein equations in vacuum**.

Hamiltonian description of the self-dual action back

- We use the **GNH approach** and **pullback everything to the primary constraint submanifold** M_0 in phase space spanned by the fields $e_{\mathrm{t}}, \omega_{\mathrm{t}} \in \Omega^0(\Sigma, \mathfrak{su}(2))\,;\, e, \omega \in \Omega^1(\Sigma, \mathfrak{su}(2))\,;\, \alpha_{\mathrm{t}} \in \Omega^0(\Sigma)\,;\, \alpha \in \Omega^1(\Sigma)$.
- **Vector fields** in M_0 have components $\mathrm{Y}_{e_{\mathrm{t}}}, \mathrm{Y}_{\omega_{\mathrm{t}}} \in \Omega^0(\Sigma, \mathfrak{su}(2)); \mathrm{Y}_e, \mathrm{Y}_\omega \in \Omega^1(\Sigma, \mathfrak{su}(2)); \mathrm{Y}_{\alpha_{\mathrm{t}}} \in \Omega^0(\Sigma); \mathrm{Y}_\alpha \in \Omega^1(\Sigma)$.
- The **presymplectic 2-form** acting on vector fields Y, Z in M_0

$$\omega(\mathrm{Z},\mathrm{Y}) = \int_\Sigma \Big(\langle \mathrm{Y}_e \,\dot\wedge\, [e \,\dot\wedge\, \mathrm{Z}_\omega] \rangle - \alpha \wedge \langle \mathrm{Y}_e \,\dot\wedge\, \mathrm{Z}_\omega \rangle - \mathrm{Z}_\alpha \wedge \langle \mathrm{Y}_\omega \,\dot\wedge\, e \rangle$$
$$- \langle \mathrm{Y}_\omega \,\dot\wedge\, [\mathrm{Z}_e \,\dot\wedge\, e] \rangle - \alpha \wedge \langle \mathrm{Y}_\omega \,\dot\wedge\, \mathrm{Z}_e \rangle + \mathrm{Y}_\alpha \wedge \langle \mathrm{Z}_\omega \,\dot\wedge\, e \rangle \Big)$$

- **Secondary constraints**

$$\alpha \wedge F + [e \,\dot\wedge\, F] = 0$$
$$D([e \,\dot\wedge\, e] + 2e \wedge \alpha) = 0$$
$$\langle e \,\dot\wedge\, F \rangle = 0$$

Hamiltonian description of the self-dual action back

- **Equations for the components** of the Hamiltonian vector field Z

$$[e \,\dot\wedge\, \mathrm{Z}_\omega] + \alpha \wedge (\mathrm{Z}_\omega - D\omega_\mathrm{t}) = \alpha_\mathrm{t} F + [e_\mathrm{t} \,\dot\wedge\, F]$$
$$[e \,\dot\wedge\, (\mathrm{Z}_e - De_\mathrm{t} - [e \,\dot\wedge\, \omega_\mathrm{t}])] + e \wedge (\mathrm{Z}_\alpha - \mathrm{d}\alpha_\mathrm{t}) = e_\mathrm{t}\mathrm{d}\alpha + [e_\mathrm{t} \,\dot\wedge\, De] - \alpha_\mathrm{t} De$$
$$\langle e \,\dot\wedge\, (\mathrm{Z}_\omega - D\omega_\mathrm{t})\rangle = \langle e_\mathrm{t} \,\dot\wedge\, F\rangle$$

- **No conditions** on $\mathrm{Z}_{e_\mathrm{t}}$, $\mathrm{Z}_{\omega_\mathrm{t}}$ and $\mathrm{Z}_{\alpha_\mathrm{t}}$. They are **arbitrary** and, hence, the dynamics of e_t^i, ω_t^i and α_t is also arbitrary.
- **Tangency conditions**

$$[\mathrm{Z}_e \,\dot\wedge\, F] + \mathrm{Z}_\alpha \,\dot\wedge\, F + [e \,\dot\wedge\, D\mathrm{Z}_\omega] + \alpha \wedge D\mathrm{Z}_\omega = 0$$
$$D\big([e \,\dot\wedge\, \mathrm{Z}_e] - \mathrm{Z}_\alpha \wedge e - \alpha \wedge \mathrm{Z}_e\big) + e \wedge \langle e \,\dot\wedge\, \mathrm{Z}_\omega\rangle + \alpha \wedge [\mathrm{Z}_\omega \,\dot\wedge\, e] = 0$$
$$\langle \mathrm{Z}_e \,\dot\wedge\, F\rangle + \langle e \,\dot\wedge\, D\mathrm{Z}_\omega\rangle = 0$$

Comments:

- One has to **solve for the vector field** in the equations written above.
- These are **linear, inhomogeneous equations**. It is important to find the simplest way to write down their solutions in order to check, later, that they **satisfy the tangency conditions**.
- **This last step is highly non-trivial**, but it is **a crucial consistency condition** that has been neglected in previous work on this subject.
- The form of ω suggests to introduce $\mathsf{H} \in \Omega^2(\Sigma, \mathfrak{su}(2))$ defined as

$$\boxed{\mathsf{H} := \frac{1}{2}[e \dot{\wedge} e] + e \wedge \alpha}$$

which would be (essentially) **canonically conjugate** to ω.

What happens if we pullback everything to M_0 and work with H, ω?

Ashtekar formulation without gauge fixing

back

- Introduce a **fiducial volume form** vol_0 on Σ and define the **vector field**

$$\widetilde{\mathrm{H}} := \left(\frac{\cdot \wedge \mathrm{H}}{\mathrm{vol}_0} \right)$$

canonically conjugate to ω in the standard sense.

- In terms of $\widetilde{\mathrm{H}}$ and ω the constraints become

$$\begin{aligned}
&\mathrm{div}_0\widetilde{\mathrm{H}} + [\omega \lrcorner \widetilde{\mathrm{H}}\,] = 0\\
&\langle \widetilde{\mathrm{H}} \lrcorner F \rangle = 0\\
&[\widetilde{\mathrm{H}} \lrcorner [\widetilde{\mathrm{H}} \lrcorner F]] = 0
\end{aligned}$$

which are the **Gauss law**, the **vector** and the **Hamiltonian constraint** of the **Ashtekar formulation** for Euclidean GR.

- When the equations for the components of the Hamiltonian vector fields are solved they give the expected dynamics.

Comments:

- **No gauge fixing** is needed! (it is not necessary to use the **time gauge**).
- The **dynamics** of Euclidean GR [in particular its full set of **symmetries**, including $SU(2) \otimes SU(2)$] is reflected in the Hamiltonian vector fields.
- The local parameters in the gauge transformations are functions of the arbitrary objects α, e_{t} and α_{t}.
- The (arbitrary) field α disappears.

Comments:

- **No gauge fixing** is needed! (it is not necessary to use the **time gauge**).
- The **dynamics** of Euclidean GR [in particular its full set of **symmetries**, including $SU(2) \otimes SU(2)$] is reflected in the Hamiltonian vector fields.
- The local parameters in the gauge transformations are functions of the arbitrary objects α, e_{t} and α_{t}.
- The (arbitrary) field α disappears.

Thank you!

... and thank you, Jurek

back