Black Hole Thermodynamics, Arbitrarily Far From Equilibrium

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I have benefitted from discussions with
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Dedicated to the memory of Jurek Lewandowski; Mentee, Colleague and Dear Friend.

Dedicated to Jurek:

Spirit Behind the Polish GRG Society

Dear Friend & Inspiring Colleague

Results I will present were deeply influenced by
Jurek's seminal contributions to Quasi-Local Horizons.

Preamble

- Einstein's equations imply that quantities defined at black hole horizons satisfy a multitude of interesting relations. Fifty years ago, Bardeen, Carter and Hawking (BCH) showed that, among relations that govern properties of nearby stationary axisymmetric black holes, there is one that has an uncanny resemblance to the first law of thermodynamics that relates the properties of nearby equilibrium states.

- Black holes out of equilibrium are described by Dynamical Horizons (DHs). Twenty years ago it was shown that Einstein equations also imply that fields on DHs also satisfy a multitude of interesting relations. New observations are:
(i) Evolution along a DH naturally defines a trajectory in the space of equilibrium states; and,
(ii) When thermodynamic parameters of equilibrium staters are transported back to MTSs of a DH using this identification, one obtains a natural generalization of the first law to black holes, arbitrarily far from equilibrium. Moreover this is a 'physical process' version.

- If one restricts oneself to infinitesimally separated MTSs, one recovers the standard first law. Moreover, this discussion removes the apparent mystery behind a recent finding that when first order perturbations are included, entropy is naturally associated with the area of an MTS that lies 'behind' the event horizon of the stationary black hole.

Plan of the talk

1. Thermodynamics of BHs in Equilibrium
2. Dynamical Horizons
3. To Non-equilibrium Thermodynamics: Strategy
4. Dynamical BHs with J=0
5. General Dynamical BHs
6. Summary and Discussion

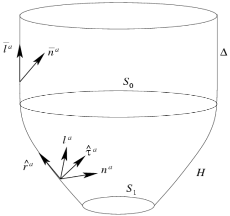
1. Thermodynamics of BHs in Equilibrium

• In vacuum GR, BHs in equilibrium are described by Kerr solutions that have two Killing vectors: t^a, φ^a. A linear combination $\bar{\ell}^a = t^a + \Omega_H \varphi^a$ is null on the Event Horizon $\equiv$ Killing Horizon. For two nearby equilibrium states we have the first law: $\delta E_H^{(t)} = \frac{\kappa_H}{8\pi G}\, \delta A_H + \Omega_H\, \delta J_H^{(\varphi)}$. $(\kappa_H \sim T;\ A_H \sim S;\ \Omega_H \sim \mu;\ J_H^{(\varphi)} \sim N)$.
Here, κ_H is the surface gravity of $\bar{\ell}^a$: $\bar{\ell}^a \nabla_a \bar{\ell}^b = \kappa_H \bar{\ell}^b$, that scales linearly with $\bar{\ell}^a$. The rescaling freedom in $\bar{\ell}^a$ is removed by requiring that t^a be unit at infinity. This strategy makes it seem that the first law is not intrinsic to the horizon.

• Remedy:
First note that φ^a has no rescaling freedom at $\mathcal{H}$ (its affine parameter $\phi \in [0, 2\pi)$). So J_H is well-defined knowing just fields on H. Next, the horizon area $A_H =: 4\pi R^2$ is also intrinsically well defined at H. So the Kerr horizons is characterized by the intrinsically defined pair R, J. Using them one can fix the rescaling freedom in $\bar{\ell}^a$ intrinsically on H by demanding that its surface gravity be $\kappa_H = \frac{(R^4 - 4G^2 J^2)}{2R^3 (R^4 + G^2 J^2)^{\frac{1}{2}}}$. Similarly, $\Omega_H = 2J\, R(R^4 + 4G^2 J^2)^{\frac{1}{2}}$.
Once $\bar{\ell}^a$ is thus fixed intrinsically, we can define t^a on H by $t^a = \bar{\ell}^a - \Omega_H \varphi^a$.

• This shift from regarding t^a as primary, and labelling equilibrium states by $(E_H^{(t)} \equiv M,\, J)$ to labeling them by the "extensive variables" (R, J) is essential in dynamical situations, e.g. with multiple BHs, because they are well-defined on each horizon in its own right, without having to refer to infinity.

2. BHs far from Equilibrium



- BH boundary now represented by a Dynamical Horizon Segment (DHS) $\mathcal{H}$; a 3-manifold, $S^2 \times R$, which is: (i) nowhere null; and, (ii) foliated by Marginally Trapped Surfaces (MTSs) S (on which $\Theta_{(\ell)} = 0$), on which the expansion $\Theta_{(n)}$ of the other null normal is nowhere vanishing.. (AA, Krishnan 2025)

For simplicity I will focus on the more common space-like $\mathcal{H}$ with $\Theta_{(n)} < 0$, and assume that $\mathcal{H}$ approaches equilibrium in the distant future (or past), represented by a Kerr isolated horizon Δ.

- Physics on $\mathcal{H}$ is governed by the constraint equations of GR (possibly with Λ):

$$C_{\mathbf{s}} := \mathcal{R} + K^2 - K_{ab}K^{ab} - 16\pi G\, T_{ab}\, \hat{\tau}^a \hat{\tau}^b = 0.$$
$$C_{\mathbf{v}}^{a} := 2D_b(K^{ab} - 2Kq^{ab}) - 16\pi G\, T_{bc}\hat{r}^b q^{ac} = 0.$$

For now, let us just note that there is a canonical procedure to drag the rotational Killing field φ^a from the Kerr IHS to $\mathcal{H}$ (AA,Campiglia,Shah). Using it in the vector constraint, one defines the angular momentum charge: $J_S^{(\varphi)} := -\frac{1}{8\pi G} \oint_S K_{ab}\varphi^a \mathrm{dS^b}$ which satisfies the balance law

$$J_{S_2}^{(\varphi)} - J_{S_1}^{(\varphi)} = -\int_{\Delta\mathcal{H}} [T_{ab}\hat{\tau}^a\varphi^b + (K^{ab} - Kq^{ab})\mathcal{L}_\varphi q_{ab}]\, \mathrm{d^3V}.$$

Thus, each MTS S has a well defined area radius R_S (so that $A_S = 4\pi R_S^2$) and $J_S^{(\varphi)}$.

3. To non-Equilibrium Thermodynamics: Strategy

• Dynamical BHS are like Open systems in which energy, angular momentum, & 'heat' can flow in. In thermodynamics, the non-equilibrium evolution is well-defined; there is a Hamiltonian. But one no longer has (global) thermodynamic parameters like Temperature, Pressure, ... This creates an immediate obstacle to extending Thermodynamics to non-equilibrium situations.

• BHs are both simpler and more complicated. Simpler because:
Each stable Equilibrium state is completely characterized by just 2-parameters; the textbook labels M, a or, more appropriate local, horizon labels R, J. Hence the space $\mathcal{E}$ of equilibrium states of a BH horizon is just 2 -dimensional!
(ii) A non-equilibrium state, by contrast, is characterized by fields on a MTS S of $\mathcal{H}$ MTS (the 2-metric; rotational 1-form, shears). Hence the space $\mathcal{NE}$ of non-equilibrium states is infinite dimensional. Yet, as we just saw, among these labels there are the two, R, J, that characterize the equilibrium states.

• Hence, unlike in the familiar thermodynamics systems, one has a natural map:

$$\pi: \mathcal{NE} \longrightarrow \mathcal{E}$$

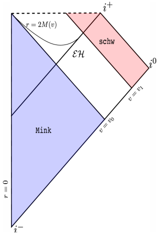
obtained by ignoring the additional rich information/micro-structure that distinguishes a MTS on one DHS and another, carrying same R, J. This is a major simplification.

• Thus, each DHS provides us with a trajectory in the space $\mathcal{E}$ of equilibrium states. Using pull-back by π, we can associate each MTS on a given DHS –i.e., a non-equilibrium state– 'instantaneous thermodynamical parameters' κ, Ω that are specific functions of R, J on $\mathcal{E}$! Under time evolution, they change because we have an open system.

• Complication: In thermodynamical systems, the background space-time provides the notion of time translation to which the Hamiltonian refers and we have an unambiguous notion of energy. For black holes in equilibrium, we could start with the Killing field $\bar{\ell}^a$, that is null and tangential to the horizon. The only freedom is in its rescaling which we fix by asking that its acceleration equal $\kappa(R, J)$. But DHS is space-like and there is no null vector tangential to it. So: What is the appropriate analog of $\bar{\ell}^a$ on a DHS?

• Let us analyze the simplest situation: Vaidya DHS.

Vaidya solution



Here, $\mathrm{ds}^2 = -(1 - \frac{2\mathrm{GM(v)}}{\mathrm{R}})\mathrm{dv}^2 + 2\mathrm{dRdv} + \mathrm{R}^2\mathrm{d}\mathring{\mathrm{s}}_2^2$. In the Schwarzschild region, $V^a \equiv \partial/\partial v$ is the static Killing field that coincides with $\bar{\ell}^a$ on the horizon, with the correct $\kappa \equiv \kappa(R, 0) = 1/2R$. On the dynamical region, $V^a = |DR|\ell^a$, is thus a natural extension of the properly normalized Killing field $\bar{\ell}^a$. This is true for all spherically symmetric DHSs. So we have a candidate for the analog of $\bar{\ell}^a$ and $E^{(V)}$ can serve as the analog of $E^{(\bar{\ell})}$.

Plan of the talk

4. Dynamical BHs with J=0

- BCH derived the first law using an appropriate linear combination of constraints on a partial Cauchy surface, with lapse N and shift N^a tailored to $\bar{\ell}^a$. For non-rotating DHSs, we have a natural extension $V^a = |DR|\ell^a$ of $\bar{\ell}^a$. Since $\ell^a = \hat{\tau}^a + \hat{r}^a$, as a first step, let us smear the constraints on $\mathcal{H}$ with $N = |DR|$ and $N^a = |DR|\hat{r}^a$ to find charges, fluxes and an energy balance law, that will be directly useful in the next slide.

- Using the fact that $\Theta_{(\ell)} = 0$ on $\mathcal{H}$ and expressing the extrinsic curvature terms in terms of projections $\sigma_\ell^{ab} = \tilde{q}^{ac}\,\tilde{q}^{bd}\,\nabla_c \ell_d$ and $\zeta^a_{(\ell)} = \tilde{q}^{ab}\,\hat{r}^c \nabla_c \ell_b$, in to MTSs, one finds

$$E^{(V)}_{S_2} - E^{(V)}_{S_1} = \int_{\Delta H} \Big\{ \underbrace{[T_{ab} V^a \hat{\tau}^b]}_{\text{matter-flux}} + \underbrace{(16\pi G)^{-1} \left[|DR| \left(|\sigma_{(\ell)}|^2 + |\zeta_{(\ell)}|^2\right)\right]}_{\text{gw-flux}} \Big\}\, \mathrm{d}^3\mathrm{V}$$

- For the $J = 0$ DHSs, the 'charge' $E_S^{(V)}$ has a direct interpretation in terms of equilibrium states $\mathcal{E}$. It equals the pull-back of $E_H^{(\bar{\ell})}$ –which is also the Schwarzschild horizon mass since $J = 0$. Thus, the right side, defined on the DHS, can also be interpreted as the change in the horizon mass along the trajectory in $\mathcal{E}$ that the map π assigns to any given DHS.

- Note that there is a gravitational wave contribution to $\Delta E^{(V)}$ even though $J = 0$. This is because, unlike for BHs in equilibrium, DHSs with $J = 0$ need not be spherically symmetric. Example: The DHSs formed during the a head-on collision of spinless black holes. Now the DHSs are not spherical and their area grows due to infalling gravitational waves! In this case, only the second term contributes.

1st law in the space of $J=0$ non-equilibrium states

- Constraint equations of GR also imply an area balance law. By multiplying the previous choice of lapse and shift by $4\pi R$, one obtains:

$$\frac{A_{S_2}-A_{S_1}}{8\pi G} = \int_{\Delta H} (\mathring{\kappa}^{-1}) \left\{ \left[T_{ab} V^a \, \hat{\tau}^b \right] + (16\pi G)^{-1} \left[|DR| \left(|\sigma_{(\ell)}|^2 + |\zeta_{(\ell)}|^2 \right) \right] \right\} \mathrm{d}^3\mathrm{V}.$$

Thus, area is the horizon 'charge' for the vector field $4\pi R V^a$, so that the 'entropy-flux' is given by the right side. The term in the curly brackets is just the energy-flux $\Delta E^{(V)}$ on $\Delta\mathcal{H}$. In the area balance law, it is multiplied by $(\mathring{\kappa})^{-1}$, where $\mathring{\kappa} = \frac{(R^4-4G^2J^2)}{2R^3(R^4+G^2J^2)^{\frac{1}{2}}} \,|_{J=0} = \frac{1}{2R}$ is the 'instantaneous temperature' assigned to any MTS of the DHS, via pull-back from $\mathcal{E}$. Hence, when S_2 and S_1 are infinitesimally close to one another, we obtain the familiar first law:

$$\frac{\mathring{\kappa}}{8\pi G}\,\delta A = \delta E^{(V)} \qquad \text{for } J=0.$$

Note that on the space of equilibrium states $\mathcal{E}$ the Killing field $\bar{\ell}^a$ –the null generator of the horizon– coincides with the time translation Killing field t^a. Hence $E^{(V)}$ on the DHS equals $E^{(\bar{\ell})} = E^{(t)} = M$ on $\mathcal{E}$.

- Note that the non-equilibrium version is a very non-trivial and subtle generalization where the 'dynamical temperature' appears inside the integral, multiplying the flux-density of energy.

5. Extension to Dynamical BHs with $J \neq 0$.

The vector field $V^a = |DR|\ell^a$ tends to $\bar{\ell}^a$ in the distant future with surface gravity $\mathring{\kappa} = \kappa(R, J = 0)$. Since $\kappa(R, J)$ scales linearly with the vector field, in the $J \neq 0$ case it is natural to use the rescaled vector field $\xi^a = \big(\kappa(R,J)/\kappa(R,0)\big)V^a \equiv (\kappa/\mathring{\kappa})\ V^a$, i.e., rescale the previous lapse and shift by $(\kappa/\mathring{\kappa})$. With this change one obtains:

$$\frac{A_{S_2} - A_{S_1}}{8\pi G} = \int_{\Delta H} (\kappa^{-1})\ \Big\{\big[T_{ab}\xi^a\,\hat{\tau}^b\big] + (16\pi G)^{-1}\big[|DR|\,(|\sigma_{(\ell)}|^2 + |\zeta_{(\ell)}|^2)\big]\Big\}\,\mathrm{d}^3\mathrm{V}.$$

where κ_H is the surface gravity of the (Kerr) equilibrium state in $\mathcal{E}$ that is the image of the MTS S under the projector π.

- The projection map $\pi : \mathcal{NE} \to \mathcal{E}$ suggests that we define a vector field t^a on the DHS $\mathcal{H}$ via $t^a = \xi^a - \Omega_H\varphi^a$ and interpret the charge (or flux) associated with φ^a as angular momentum charge (or flux) and that associated with t^a as the charge (or flux) of t-energy. Then, setting $\xi^a = t^a + \Omega_H\varphi^a$) and using the angular momentum balance law from part 2, one obtains:

$$\begin{aligned}\frac{A_{S_2} - A_{S_1}}{8\pi G} &= \int_{\Delta H} (\kappa^{-1})\Big\{\big[\mathcal{F}^{(t)}_{\mathrm{matt}} + \mathcal{F}^{(t)}_{\mathrm{gw}}\big] + \Omega\big[\mathcal{F}^{(\varphi)}_{\mathrm{matt}} + \mathcal{F}^{(\varphi)}_{\mathrm{gw}}\big]\Big\}\,\mathrm{d}^3\mathrm{V}\\ &= \int_{\Delta H} (\kappa^{-1})\Big\{\big[\mathcal{F}^{(t)}_{\mathrm{matt}} + \mathcal{F}^{(t)}_{\mathrm{gw}}\big] - \big[(\oint_{S_2} - \oint_{S_1})\ \Omega\, j^{\varphi} d^2V\big]\end{aligned}$$

- Hence, when S_2 and S_1 are separated infinitesimally, one has the familiar 1st law: $\frac{\kappa}{8\pi G}\,\delta A = \delta E^{(t)} + \Omega_H \delta J^{(\varphi)}$. But on DHS κ, Ω are the 'instantaneous' surface gravity and angular velocity, associated with each MTS. Again, the fact that they appear inside the integral is a non-trivial feature of dynamical BHs.

6. Summary and Discussion.

• In both equilibrium and non-equilibrium situations, the first law is a direct consequence of the constraint equations of GR in presence of horizons. There are infinitely many identities implied by constraints. The first law emerges when a judicious choice of lapse N and Shift N^a is used to smear constraints. (In the original BCH analysis, N, N^a used in the constraints on a partial Cauchy slice correspond to the correctly 'normalized' Killing vector $\bar{\ell}^a$ that is the null generator of the horizon.) In the dynamical situation N, N^a correspond to the null normal ξ^a to MTSs of $\mathcal{H}$ that tends $\bar{\ell}^a$ as equilibrium is approached.

• An extension to fully non-equilibrium situations is possible because of two non-trivial features of BHs in GR –discussed in part 3– that enables one to unambiguously assign the 'thermodynamic parameters' to generic non-equilibrium states. The result is an physical process version of the first law, in contrast to the pioneering BCH analysis.

• What I presented is 'an' extension of the equilibrium thermodynamics to generic non-equilibrium situations. It is very natural, but other extensions are possible. For example, one may come up with another way to define φ^a on a DHS and then the map $\pi : \mathcal{NE} \rightarrow \mathcal{E}$ will change but one may still obtain an acceptable generalization of the first law to dynamic BHs.

Comparision and Suggestions

- In the case of BH mergers, the framework works for the DHS of each progenitor as well as of the remnant. These situations cannot be encompassed by any of the perturbative approaches proposed since they only consider perturbations on stationary space-times with an event horizon. Also, in the one proposed by Hollands, Wald & Zhang one begins with the event/Killing horizon of a stationary BH and arrives at a surprise that the perturbatively 'corrected' 1st law refers to the area of an MTS inside the EH. This is 'explained' by the fact that in a fully dynamical situation, the first law always refers to the area of a MTS on a DHS, which is indeed inside the EH!

- There is a recent approach to non-equilibrium statistical mechanics put forward by Buča in which the non-equilibrium evolution is also considered as a trajectory in the space of equilibrium (Gibbs) states. It would be interesting to explore the similarities and differences as they may suggest fertile directions for us.

- This setting –in which interesting physics arises from constraints on a space-like surface– seems well-suited to quantum generalization via LQG techniques. In particular, the linear combination of constraints that features is very natural in a spinorial framework: $(o^A o^{\dagger\, D})\,(F_{ab\,A}{}^B\, \sigma^{a}_{B}{}^{C}\, \sigma^{b}_{CD})$. Also, the hypersurface Twistors (that Lionel introduced) may provide new insights.